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THEORY AND EXPERIMENTAL EVALUATION OF ADAPTIVE CONTROL AND IDENTIFICATION METHODS

by

© VASSILIOS TZOUANAS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

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Date......May 10, 1983.......

To my parents and brother

Abstract

In adaptive control, the parameters that appear in the mathematical model of the process can be identified by using different identification algorithms usually on an on-line basis. Insufficient knowledge of the process parameters in an explicit control design or of the controller parameters in an implicit one can have catostrophic results on the performance of the controller or the stability of the closed loop system. Proof of global stability of a closed loop control system is not an easy task. Only a few of the proposed control algorithms have been proven to be globally stable. One of the most common assumptions in proving stability requires that the controlled system must be minimum phase.

In this thesis, a least squares identification scheme is shown to converge to the true parameter values of a stable open loop system if a persistently exciting system input and a variable fogetting factor less than one are used. A pole-assignment control algorithm is proven to be globally stable for both minimum and nonminimum phase systems despite the presence of bounded disturbances. The use of a persistently exciting signal is essential in proving stability for the closed loop system controlled by the pole-assignment algorithm. The problem of large process time-delays is handled by using a Smith predictor. For a particular class of processes, this pole-assignment algorithm is shown to take the form of a three term velocity

type adaptive PID controller. Simulation and experimental results on a batch solution methylmethacrylate polymerizer demonstrate the usefulness of this pole-assignment control algorithm.

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1. Introduction and objectives

This chapter is provided as a guide to the variety of topics covered in this thesis since the subject being treated requires knowledge in the areas of process identification and parameter estimation, adaptive control, polymer chemistry and engineering, as well as process computer control. However, the ultimate objective of this study is to experimentally evaluate the performance of an adaptive control algorithm which has been proven to be globally stable. The controller performance is evaluated by controlling the monomer conversion in a batch polymerization reactor of methylmethacrylate.

The conventional three term PID controller is the most commonly used control algorithm in chemical processes and this is likely to remain so in the foreseeable future. There are however situations where this controller does not perform satisfactorily. Processes with highly non-linear dynamics, time-varying characteristics and strong interaction between the state variables are difficult to control by the fixed gain PID controller or even a controller with gain scheduling. In such cases, periodic retuning of the controller gains is needed. Instead of having skilled operators do the periodic tuning of the controller, it is a better idea to use a controller which can tune or adapt itself as the process dictates.

In adaptive control, the mathematical representation of the controlled process is of prime importance. It is very well known that the exact process model is not always available or if it is, it may be so complex that its usefulness is limited. In such cases, a model must be assumed. The assumed model must combine all the knowledge about the process in a usuable form but with a minimum degree of complexity. The different coefficients appearing in the mathematical model of the process are experimentally identified based on measurements taken from the process.

Since its early introduction, the area of adaptive control has grown rapidly and there are a number of different adaptive control algorithms proposed in the literature. Most of these control algorithms have been or were initially proposed without any proof of stability of the closed loop system. It was not until recently that stability results have appeared for a restricted number of adaptive control algorithms. Some of the proposed control algorithms involve pole-zero cancellation. This has given rise to a misapprehension that nonminimum phase systems can pose insoluble problems with the application of adaptive control. Pole assignment control algorithms are a particular class of adaptive control algorithms which partly overcome this problem by shifting the closed loop poles to desired locations and can conveniently handle nonminimum phase systems.

It is the objective of this thesis to study the scope and significance of persistently exciting signals in adaptive system identification and control as well as a particular type of pole-assignment algorithm, understand its limitations, impose sufficient conditions for the global stability of a closed loop system under its control and evaluate the performance of the resulting algorithm by simulations as well as experimental applications. This thesis has been divided into six chapters.

In Chapter 2, the identification scheme that will be used in this work is presented. This identification scheme is a variant of the least squares algorithm originally derived by Gauss. Different forms for the original least squares algorithm are presented. The need for using a variable forgetting factor is also discussed. The usefulness of the indentification algorithm used in this work is established by proving convergence properties for the prediction error and the parameters estimates. It is also demonstrated that the use of a persistently exciting signal and a variable forgetting factor less than one are sufficient conditions for the converge of the parameter estimates to their true values.

In Chapter 3, an adaptive pole-assignment control algorithm is discussed. Local stability results requiring good apriori knowledge about the true process parameters are derived for both minimum and nonminimum phase systems despite the presence of bounded disturbances. These local stability results are extended to global stability results if a fixed gain controller able to stabilize the system and a persistently exciting reference signal are used.

In Chapter 4, the adaptive pole-assignment control algorithm presented in the previous chapter is modified in order to easily handle large time delays. The idea is to use a Smith predictor for dead time compensation. By including a Smith predictor in the closed loop system, the time delay in its characteristic polynomial is cancelled. This makes the control algorithm computationally more efficient. It is also shown how the pole-assignment algorithm with a Smith predictor can take the form of a three term velocity type adaptive PID controller for a particular class of processes.

In Chapter 5, the performance of the pole-assignment algorithm is evaluated both experimentally and by simulation studies. The system under control is a batch reactor for the solution polymerization of methylmethacrylate. We are mainly interested in controlling the monomer conversion within the reactor. Weight average molecular weight control is demonstrated by simulation studies. The batch reactor is also controlled by using a fixed gain PID controller and a self-tuning controller. The results of these two algorithms are used for the comparison of the control performance of the adaptive pole-assignment control algorithm.

The final summary and overall conclusions of this thesis are discussed in Chapter 6 followed by suggestions for future extensions to this study.

2. Identification methods

2.1 Introduction

An adaptive controller adjusts its parameters on the basis of information obtained during the control period so as to accommodate for changes in system parameters due to operating condition or environmental changes. A controller, designed to control a linearized perturbation model of a process, will not perform well as the steady state changes due to changes in the process and operating conditions. Also, the nonstationary nature of many processes is another reason for using adaptive control.

Although the reasons for using adaptive control were obvious, the first adaptive control systems were not introduced until the late 1950's. The primary objective was to design adaptive controllers for use in autopilots to improve the performance of aircraft over a wide variety of operating conditions. The very first attempts in establishing powerful and useful controllers were unsuccessful due to the poor hardware and the lack of adequate theory.

It was not until 1973, when Aström and Wittenmark (1973) presented the first landmark paper on self-tuning regulators. This paper revived the interest in adaptive control. Since then, due also to the rapid progress in computer technology and to the already developed control theory during 1960's, a number of adaptive controllers have been introduced (Landau (1973), Martin-Sanchez (1974), Monopoli (1974). Clarke and Gawthrop (1975), Fewer and Morse (1978), Goodwin and Ramadge (1979), Narendra and Lin (1980) to name a few). Also, some successful applications of adaptive control have been reported (Cergell and Hedqvist (1974), Borisson and Wittenmark (1975), Seborg and Fisher (1977), Dumont and Belanger (1978), Johnstone et. al. (1979), Cergell and Hedqvist (1974), Borisson and Syding (1976), Jutan, Wright and MacGregor (1984), Morris and Nazer (1977), Landau and Coutriol (1972), Martin-Sanchez and Shah (1984) to mention a few).

In most cases, the design of an adaptive controller is based on the ideas of the Certainty Equivalence Principle (Aström (1970)). According to this principle, the controller is designed as if the process parameters were known. Since we do not know the exact process parameters we use, in the control law, estimates of them. So it becomes clear that "good" estimates of the process parameters is a prerequisite for good control performance. In other words the process of parameter estimation is very important in adaptive control. Because of it, a lot of effort has been devoted in this area during the last two decades. The result of this effort is a large number of papers specifically discussing various aspects of the parameter estimation on line problem (Eykhoff (1974), Landau (1976), Lung (1977), (1981); Isermann (1981), Kumar (1983) to name a few).

Parameter estimation is defined by Eykoff (1974) as the experimental determination of values of parameters that govern the dynamic and/or non-linear behavior, assuming that the structure of the process model is known. The estimation procedure can be performed off-line or on-line. The off-line estimation requires a priori collection of input/output data and gives estimates with higher precision (Aström and Wittenmark (1984)). The on-line estimation gives the estimates as the measurements are available and is the method that is used in practice.

Parameter estimation can be done only if a means of mathematical description of the process is available. This introduces the idea of a model. The model must describe the essential characteristics of a process and provide useful information about it. This is very important since further control decisions will be based on this information. There is no need for the model to be complex. The degree of complexity must vary on the way that we use it. A model can be obtained by applying the laws of physics, chemistry or suitable conservation balances. This approach of model building is not always easy. It requires a deeper understanding of the process which is not always available. Sometimes this understanding cannot be expressed in a mathematical way or if it can be expressed, the complexity of the resulting model is such that its usefulness becomes questionable. A particular class of models, commonly used in adaptive control is that of Autoregressive Moving Average

(ARMA) models. Before we proceed with the estimation of the parameters, the structure of the model must be defined or assumed. Also the experimental conditions, under which the estimation will be done, must be determined. Open-loop conditions should be preferred but identification under closed-loop conditions is usually the case. In the latter case, by keeping the whole system close to a desired state we restrict the input/output signals. These signals will not contain enough information or will not be able to excite all the system modes. Excitation of the system modes is a requirement for parameter estimation.

Since the process parameters are not known, the judgement of the estimated parameters is based on some criteria. Depending on the criteria, different adaptation laws can be derived. For instance, a criterion can define the current best estimate of the true parameter vector as that vector which is closest to the previous parameter vector. By satisfying this criterion, we are led to the Projection algorithm. By minimizing the sum of the squares of the prediction errors, the least squares algorithm results. It must be mentioned that the prediction error is defined as the difference between the process output and the estimated model output. In this case, both the process and the model are subjected to the input signal but only the process output is corrupted by noise, if present.

2.2 The least squares algorithm

One of the most frequently used identification algorithms is the least squares algorithm. This section is devoted to a description of this algorithm.

Consider a discrete time, single input/single output, linear dynamic system described by a Deterministic Autoregressive Moving Average (DARMA) model as

$$A(z^{-1})\ y(k) = B(z^{-1})\ u(k) \tag{2.1}$$

where

$$A(z^{-1}) = 1 - a_1 z^{-1} - \dots - a_n z^{-n} \tag{2.2}$$

$$B(z^{-1}) = b_{d+1} z^{-d-1} + \dots + b_{d+m} z^{-(d+m)} \tag{2.3}$$

u(k) and y(k) are the input and the output signals at the discrete time instant k; d is the time delay ($d \geq 0$).

The system (2.1) can be written in a compact vector form as

$$y(k) = \underline{\psi}(k-1)'\underline{\theta}_0 \tag{2.5}$$

where

$$\underline{\psi}(k-1)' = [y(k-1) \dots y(k-n)\ u(k-d-1) \dots u(k-d-m)] \tag{2.6}$$

$$\underline{\theta}_0' = [a_1 \dots a_n\ b_{d+1} \dots b_{d+m}] \tag{2.7}$$

Assume that a model approximating the actual process is described by

$$\hat{y}(k) = \underline{\psi}(k-1)^t \underline{\theta}(k-1) \tag{2.8}$$

where $\hat{y}(k)$ is an estimate of the process output when the estimate, $\underline{\theta}(k-1)$, of the true parameter vector, $\underline{\theta}_o$, is used. The vector $\underline{\theta}(k-1)$ is given by

$$\underline{\theta}(k-1)^t = [\hat{a}_1(k-1) \ \ldots \ \hat{a}_n(k-1) \ \hat{b}_{d+1}(k-1) \ \ldots \ \hat{b}_{d+m}(k-1)] \tag{2.9}$$

The prediction error e(k) is defined by

$$e(k) = y(k)-\hat{y}(k) \tag{2.10}$$

As it was previously mentioned, in least squares the sum of the squares of the prediction errors must be minimum. In other words, the loss function

$$J_k(\underline{\theta}) = \frac{1}{2} \sum_{i=1}^{k} e(i)^2 \tag{2.11}$$

must be minimal.

Instead of minimizing the loss function (2.11) we can minimize the following function

$$J_k(\underline{\theta}) = \frac{1}{2} \sum_{i=1}^{k} e(i)^2 + \frac{1}{2} [\underline{\theta}-\underline{\theta}(0)]^t \, \underline{P}(0)^{-1} \, [\underline{\theta}-\underline{\theta}(0)] \tag{2.12}$$

By minimizing the loss function (2.12), a penalty is paid for the initial guess, $\underline{\theta}(0)$, of the true parameter vector $\underline{\theta}_0$. The matrix $\underline{P}(0)$ is a weighting factor for the guess $\underline{\theta}(0)$. It reflects our confidence about the initial guess. Large values for the elements of $\underline{P}(0)$ correspond to a poor guess while small values correspond to a good initial guess. The algorithm which minimizes the loss function (2.12) is given by

$$\underline{\theta}(k) = \underline{\theta}(k-1) + \underline{K}(k)\ e(k) \tag{2.13}$$

$$\underline{K}(k) = \frac{\underline{P}(k-1)\ \underline{\psi}(k-1)}{1 + \underline{\psi}(k-1)'\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \tag{2.14}$$

$$\underline{P}(k) = \underline{P}(k-1) - \underline{K}(k)\ \underline{\psi}(k-1)'\underline{P}(k-1) \tag{2.15}$$

where $\underline{K}(k)$ is the estimator gain vector.

In the minimization of the loss function (2.12) all the available data are taken into account. At least this is not desirable for a time varying process. Since old data do not correspond to the present state of a time varying process, they must be discounted. By minimizing the loss function

$$J_k(\underline{\theta}) = \frac{1}{2}\sum_{i=1}^{k} \lambda^i\ e(i)^2 + \frac{1}{2}\ [\underline{\theta}-\underline{\theta}(0)]'\underline{P}(0)^{-1}[\underline{\theta}-\underline{\theta}(0)] \tag{2.16}$$

the past data are exponentially discounted if the forgetting factor λ is less than one. The speed of adaptation is determined by the asymptotic memory length (Clarke and Gawthrop (1975))

$$N = \frac{1}{1 - \lambda} \qquad (2.17)$$

which means that the past data are forgotten after N sampling intervals. In this case,

$$\underline{\theta}(k) = \underline{\theta}(k-1) + \underline{K}(k)\, e(k) \qquad (2.18)$$

$$\underline{K}(k) = \frac{\underline{P}(k-1)\, \underline{\psi}(k-1)}{\lambda + \underline{\psi}(k-1)'\, \underline{P}(k-1)\, \underline{\psi}(k-1)} \qquad (2.14)$$

$$\underline{P}(k) = \frac{1}{\lambda} \left[\underline{P}(k-1) - \underline{K}(k)\, \underline{\psi}(k-1)'\, \underline{P}(k-1)\right] \qquad (2.20)$$

The application of the exponential data discounting also is helpful for a time-invariant process during the transient state. After steady state conditions have been established, the use of constant forgetting factor, creates a problem often referred to as "blow-up". The covariance matrix grows exponentially fast and the system becomes sensitive to disturbances and susceptible to computational errors (Aström and Wittenmark (1980)). The use of a variable forgetting factor (Fortesque, Kershenbaum and Ydstie (1981)) is one way to overcome this problem. The idea is quite straightforward. During the transient state, when there is a lot of information about the system, a forgetting factor can be used. During steady-state operation, when little information is obtained about the system, the forgetting factor should go to one. In this way, the "blow up" problem is eliminated. But the next problem is to find a way of adjusting the forgetting factor. The forgetting factor is chosen in such a way that a scalar measure of the information content of the regressor is kept constant. In a mathematical form, the law of adaptation of the forgetting factor is given by

$$\lambda(k) = 1 - [1 - \underline{\psi}'(k-1)\ \underline{K}(k)]\ e(t)^2 / \Sigma_0 \qquad (2.21)$$

and

$$\lambda(k) = \lambda_{min} \text{ if } \lambda(k) \leq \lambda_{min} \qquad (2.22)$$

where Σ_0 is related to the sum of the squares of the prediction errors. It is given by:

$$\Sigma_0 = \sigma_0^2\ N_0 \qquad (2.23)$$

where σ_0^2 is the expected measurement noise variance ($\sigma_0^2=1$ if noise is not present) while N_0 corresponds to a nominal asymptotic memory length and controls the speed of adaptation. N_0 is a user defined parameter.

It is obvious from the previous discussion that different identification algorithms result, depending on the criterion under minimization. Before an identification scheme is used for parameter estimation, in a closed-loop control system, it must be sure that it guarantees some useful properties like boundness of the parameter estimates, asymptotic convergence of the parameters and minimization of the prediction error.

2.3 Stability Analysis

In this section we shall prove that the identification scheme used in this work guarantees

a. Boundness of the parameters estimates.

b. Asymptotic convergence of the parameters.

c. Minimization of the prediction error for a stable open loop process.

The identification scheme used in this work is the recursive least squares algorithm with a variable forgetting factor as proposed by Fortesque et. al. (1981). The equation that updates the estimator gain vector is slightly different than equation (2.19), instead of equation (2.19) equation (2.14) is used.

In summary, the equations used in order of execution for the open loop identification of a DARMA process are given by:

Process: $$y(k) = \underline{\psi}(k-1)' \; \underline{\theta}_o \qquad (2.24)$$

Model: $$\hat{y}(k) = \underline{\psi}(k-1)' \; \underline{\hat{\theta}}(k-1) \qquad (2.25)$$

Prediction Error: $$e(k) = y(k) - \hat{y}(k) \qquad (2.26)$$

Estimator Gain: $$\underline{K}(k) = \frac{\underline{P}(k-1)\ \underline{\psi}(k-1)}{1 + \underline{\psi}(k-1)'\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \qquad (2.27)$$

Parameter Adaptation: $$\underline{\theta}(k) = \underline{\theta}(k-1) + \underline{K}(k)\ e(k) \qquad (2.28)$$

Forgetting Factor: $$\underline{\lambda}(k) = 1 - \frac{e(k)^2}{[1 + \underline{\psi}(k-1)'\underline{P}(k-1)\underline{\psi}(k-1)]\Sigma_o} \qquad (2.29)$$

or

$$\lambda(k) = \lambda min \quad \text{if} \quad \lambda(k) < \lambda min$$

with

$$\Sigma_o = \sigma_o^2\ N_o$$

Covariance Matrix: $$\underline{P}(k) = \frac{1}{\lambda(k)}\ [\underline{P}(k-1) - \underline{K}(k)\ \psi(k-1)'\ \ \underline{P}(k-1)] \qquad (2.30)$$

where λmin is a minimum value for the forgetting factor. All the other parameters have the same meaning as they were defined in the previous section.

Stability results for least squares algorithms in an open loop identification problem have been derived by different investigators. Johnstone (1982) has studied a least squares identification scheme with constant forgetting factor. Cordero and Mayne (1981) used a least squares identification algorithm with variable forgetting factor in a closed loop system. They proved stability for the closed loop system by deriving useful properties for the identification scheme. One of their assumptions was requiring an upper bound for the trace of the covariance matrix for all times. Our studies on the stability of the identification scheme (2.24) to (2.30) presented by theorem 2.1 can be considered as an extension of Johnstone's work when the forgetting factor is variable and not constant. In contrast with Cordero's work we do not impose any restrictions on the trace of the covariance matrix.

Theorem 2.1

Under the following assumptions:

A1] The process is given by equation (2.24)

A2] The system is stable (for bounded $\{u(k)\}$, it gives bounded $\{y(k)\}$).

A3] The input sequence $\{u(k)\}$ is bounded

the above mentioned identification algorithm, described by equations (2.24) to (2.30), guarantees that

P1] $$||\underline{\theta}_0-\underline{\theta}(k)|| \le ||\underline{\theta}_0-\underline{\theta}(0)|| \qquad (2.31)$$

P2] $$\lim_{k\to\infty} [\underline{\theta}(k)-\underline{\theta}(k-1)] = 0 \qquad (2.32)$$

P3] $$\lim_{k\to\infty} e(k) = 0 \qquad (2.33)$$

Proof:

Define: $\underline{\theta}(k) = \underline{\theta}_0 - \underline{\theta}(k)$ (2.34)

From equations (2.24) to (2.26) and (2.34)

$$e(k) = \underline{\psi}(k-1)^t \, \underline{\theta}(k-1) \qquad (2.35)$$

Equations (2.27), (2,28), (2.34) and (2.35) give

$$\underline{\theta}(k) = \left[I - \frac{\underline{P}(k-1)\ \underline{\psi}(k-1)\ \underline{\psi}(k-1)^t}{1 + \underline{\psi}(k-1)^t\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \right] \underline{\theta}(k-1) \qquad (2.36)$$

Also, from equations (2.27) and (2.30)

$$\lambda(k)\ \underline{P}(k)\ \underline{P}(k-1)^{-1}\ \underline{\theta}(k-1) = \left[I - \frac{\underline{P}(k-1)\ \underline{\psi}(k-1)\ \underline{\psi}(k-1)^t}{1 + \underline{\psi}(k-1)^t\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \right] . \underline{\theta}(k-1) \qquad (2.37)$$

Equations (2.36) and (2.37) give

$$\underline{P}(k)^{-1} \underline{\theta}(k) = \lambda(k)\ \underline{P}(k-1)^{-1}\ \underline{\theta}(k-1) \qquad (2.38)$$

Also,

$$\underline{P}(k)^{-1} = \lambda(k)\ [\underline{P}(k-1)^{-1} + \underline{\psi}(k-1)\ \underline{\psi}(k-1)^t] \qquad (2.39)$$

The validity of equation (2.39) can be proven by using the matrix inversion lemma (Franklin and Powell (1980)), through back substitutions. According to this lemma

$$[A+BCD]^{-1} = A^{-1} - A^{-1}\ B\ [C^{-1} + D\ A^{-1}\ B]\ D\ A^{-1} \qquad (2.40)$$

If

$$A = \underline{P}(k-1)^{-1}$$

$B = \underline{\psi}(k-1)$

$C = 1$

$D = \underline{\psi}(k-1)^{t}$

then through substitutions into equation (2.40)

$$[\underline{P}(k-1)^{-1} + \underline{\psi}(k-1)\ \underline{\psi}(k-1)^{t}]^{-1} = \underline{P}(k-1) - \underline{P}(k-1)\ \underline{\psi}(k-1) \cdot [1 + \underline{\psi}(k-1)^{t}\ \underline{P}(k-1)\ \underline{\psi}(k-1)]^{-1} \cdot \underline{\psi}(k-1)^{t}\ \underline{P}(k-1)$$

or

$$\underline{P}(k) = \frac{1}{\lambda(k)} \left[\underline{P}(k-1) - \frac{\underline{P}(k-1)\ \underline{\psi}(k-1)\ \underline{\psi}(k-1)^{t}\ \underline{P}(k-1)}{1 + \underline{\psi}(k-1)^{t}\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \right] \quad (2.40a)$$

But equation (2.40a) is the adaptation law for the covariance matrix. This proves the validity of equation (2.39).

Now define the Lyaponov type scalar function V(k) as

$$V(k) = \underline{\theta}(k)^{t}\ \underline{P}(k)^{-1}\ \underline{\theta}(k) \quad (2.41)$$

From equation (2.38)

$$\underline{\theta}(k) = \lambda(k)\ \underline{P}(k)\ \underline{P}(k-1)^{-1}\ \underline{\theta}(k-1) \quad (2.42)$$

Equations (2.41) and (2.42) give

$$\begin{aligned} V(k) - V(k-1) &= [\lambda(k)\ \underline{P}(k)\ \underline{P}(k-1)^{-1}\ \underline{\theta}(k-1)]^{t}\ \underline{P}(k)^{-1} \\ &\quad \cdot[\lambda(k)\ \underline{P}(k)\ \underline{P}(k-1)^{-1}\ \underline{\theta}(k-1)] - \\ &\quad - \underline{\theta}(k-1)^{t}\ \underline{P}(k-1)^{-1}\ \underline{\theta}(k-1) \\ &= \underline{\theta}(k-1)^{t} \Big[\{\underline{P}(k-1)^{-1}\}^{t}\ \lambda(k)\ \{\lambda(k)\ \underline{P}(k)^{t}\}\ \underline{P}(k-1)^{-1} \\ &\quad - \underline{P}(k-1)^{-1} \Big]\ \underline{\theta}(k-1) \end{aligned} \quad (2.43)$$

Equation (2.40a) gives:

$$\lambda(\kappa)\,\underline{\underline{P}}(\kappa)^{\tau} = \underline{\underline{P}}(\kappa-1)^{\tau}\left[\,\underline{\underline{I}} - \frac{\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)\,\underline{\Psi}(\kappa-1)^{\tau}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)}\right]. \qquad (2.44)$$

Equations (2.43) and (2.44) give:

$$\begin{aligned} V(\kappa) - V(\kappa-1) &= \underline{\tilde{\theta}}(\kappa-1)^{\tau}\Bigg[\{\underline{\underline{P}}(\kappa-1)^{-1}\}^{\tau}\lambda(\kappa)\,\underline{\underline{P}}(\kappa-1)^{\tau} \\ &\quad \left[\underline{\underline{I}} - \frac{\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)\,\underline{\Psi}(\kappa-1)^{\tau}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)}\right]^{\tau}\cdot\underline{\underline{P}}(\kappa-1)^{-1} - \underline{\underline{P}}(\kappa-1)^{-1}\Bigg]\,\underline{\tilde{\theta}}(\kappa-1) \\ &= \underline{\tilde{\theta}}(\kappa-1)^{\tau}\Bigg[\lambda(\kappa)\,\underline{\underline{P}}(\kappa-1)^{-1} - \frac{\lambda(\kappa)\,\underline{\Psi}(\kappa-1)\,\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)^{-1}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} \\ &\quad - \underline{\underline{P}}(\kappa-1)^{-1}\Bigg]\,\underline{\tilde{\theta}}(\kappa-1) \qquad (2.45) \end{aligned}$$

From equation (2.39) we can conclude that the covariance matrix $\underline{\underline{P}}(k)$ is a postive definite and symmetric matrix, provided that $\underline{\underline{P}}(0)$ is also. Then, equation (2.45) gives:

$$\begin{aligned} V(\kappa) - V(\kappa-1) &= \underline{\tilde{\theta}}(\kappa-1)^{\tau}\left[\{\lambda(\kappa)-1\}\,\underline{\underline{P}}(\kappa-1)^{-1} - \frac{\lambda(\kappa)\,\underline{\Psi}(\kappa-1)\,\underline{\Psi}(\kappa-1)^{\tau}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)}\right] \\ &\quad\cdot\underline{\tilde{\theta}}(\kappa-1) = \{\lambda(\kappa)-1\}\,\underline{\tilde{\theta}}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)^{-1}\,\underline{\tilde{\theta}}(\kappa-1) \\ &\quad - \frac{\lambda(\kappa)\left[\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{\tau}\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} \\ &= \{\lambda(\kappa)-1\}\,V(\kappa-1) - \frac{\lambda(\kappa)\left[\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} \end{aligned}$$

then

$$V(\kappa) = \lambda(\kappa)\, V(\kappa-1) - \frac{\lambda(\kappa)\left[\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} \qquad (2.46)$$

Equation (2.46) means that the scalar function V(k) is nonincreasing. By definition, V(k) is a nonegative function. So, for large times (k→∞) it must converge. Suppose that V, is its limiting value. Then, equation (2.46) gives:

$$\left[1-\lambda(\kappa)\right]V_c = -\frac{\lambda(\kappa)\left[\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} \qquad (2.47)$$

Since $0<\lambda(k)\leq 1$, the left hand side of equation (2.47) is positive or zero irrespective of the value of V . The right hand side of this equation can be negative or zero. It must be noted that at this point nothing is said about the convergence of λ(k) or of the right hand side term of equation (2.47). But since equation (2.47) must be valid we conclude that both sides are zero as k→∞. This automatically means that both the above mentioned terms converge. In particular,

$$\lim_{\kappa\to\infty} \frac{\lambda(\kappa)\left[\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} = 0 \qquad (2.48)$$

Since λ(k) is bounded ($0<\lambda(k)\leq 1$) equation (2.48) means that

$$\lim_{\kappa\to\infty} \frac{\left[\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\underline{\Psi}(\kappa-1)^{\tau}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} = 0 \qquad (2.49)$$

or

$$\lim_{\kappa\to\infty} \frac{e(\kappa)^2}{1 + \underline{\Psi}(\kappa-1)^\tau \, \underline{\underline{P}}(\kappa-1) \, \underline{\Psi}(\kappa-1)} = 0 \tag{2.50}$$

if equation (2.26) is taken into account.

From equations (2.29) and (2.50) we conclude that

$$\lim_{k\to\infty} \lambda(k) = 1 \tag{2.51}$$

Now define the scalar function W(k) as

$$W(k) = \{\prod_{i=1}^{k} \lambda(i)^{-1}\} \, V(k) \tag{2.52}$$

By multiplying equation (2.46) by

$$\{\prod_{i=1}^{k} \lambda(i)^{-1}\}$$

the following results

$$W(\kappa) = W(\kappa-1) - \frac{\{\prod_{i=1}^{\kappa-1} \lambda(i)^{-1}\} \left[\underline{\Psi}(\kappa-1)^\tau \, \underline{\tilde{\theta}}(\kappa-1) \right]^2}{1 + \underline{\Psi}(\kappa-1)^\tau \, \underline{\underline{P}}(\kappa-1) \, \underline{\Psi}(\kappa-1)} \tag*{(2.53}$$

Equation (2.53) means that the scalar function W(k) is nonincreasing. By definition W(k) is nonnegative. Both of these imply that W(k) will converge as k→∞ . Regardless of its limit, the following is valid

$$\lim_{\kappa\to\infty} \frac{\{\prod_{i=1}^{k-1} \lambda(i)^{-1}\} \left[\underline{\Psi}(\kappa-1)^\tau \, \underline{\tilde{\theta}}(\kappa-1) \right]^2}{1 + \underline{\Psi}(\kappa-1)^\tau \, \underline{\underline{P}}(\kappa-1) \, \underline{\Psi}(\kappa-1)} = 0 \tag{2.54}$$

Equation (3.39) implies that

$$\lambda(\kappa)\,\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-1)^{-1}\right] \leq \bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa)^{-1}\right] \qquad (2.55)$$

where $\lambda\min\ [P(k)^{-1}]$ is the minimum eigenvalue if $P(k)^{-1}$. Inequality (2.55), written at different time instants, gives

$$\lambda(\kappa-1)\,\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-2)^{-1}\right] \leq \bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-1)^{-1}\right]$$

$$\lambda(\kappa-2)\,\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-3)^{-1}\right] \leq \bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-2)^{-1}\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\lambda(1)\,\bar{\lambda}_{\min}\left[\underline{\underline{P}}(0)^{-1}\right] \leq \bar{\lambda}_{\min}\left[\underline{\underline{P}}(2)^{-1}\right]$$

The above set of inequalities implies that

$$\Big\{\prod_{i=1}^{\kappa-1}\lambda(i)\Big\}\,\bar{\lambda}_{\min}\left[\underline{\underline{P}}(0)^{-1}\right] \leq \bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-1)^{-1}\right] \qquad (2.56)$$

or that

$$\frac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-1)^{-1}\right]} \leq \Big\{\prod_{i=1}^{\kappa-1}\lambda(i)^{-1}\Big\}\,\frac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(0)^{-1}\right]} \qquad (2.57)$$

By using inequality (2.57), the following can be obtained

$$\frac{\Big\{\prod_{i=1}^{\kappa-1}\lambda(i)^{-1}\Big\}\left[\underline{\Psi}(\kappa-1)^{T}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\underline{\Psi}(\kappa-1)^{T}\,\underline{\underline{P}}(\kappa-1)\,\underline{\Psi}(\kappa-1)} \geq \frac{\Big\{\prod_{i=1}^{\kappa-1}\lambda(i)^{-1}\Big\}\left[\underline{\Psi}(\kappa-1)^{T}\,\underline{\tilde{\theta}}(\kappa-1)\right]^{2}}{1+\bar{\lambda}_{\max}\ \underline{\underline{P}}(\kappa-1)\ \ \underline{\Psi}(\kappa-1)^{T}\,\underline{\Psi}(\kappa-1)}$$

$$\geq \frac{\{\prod_{i=1}^{\kappa-1} \lambda(i)^{-1}\} \left[\underline{\Psi}(\kappa-1)^{\tau} \, \underline{\tilde{\theta}}(\kappa-1) \right]^2}{1 + \dfrac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-1)^{-1} \right]} \underline{\Psi}(\kappa-1)^{\tau} \, \underline{\Psi}(\kappa-1)}$$

$$\geq \frac{\{\prod_{i-1}^{\kappa-1} \lambda(i)^{-1}\} \left[\underline{\Psi}(\kappa-1)^{\tau} \, \underline{\tilde{\theta}}(\kappa-1) \right]^2}{1 + \{\prod_{i=1}^{\kappa-1} \lambda(i)^{-1}\} \dfrac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(0)^{-1} \right]} \underline{\Psi}(\kappa-1)^{\tau} \, \underline{\Psi}(\kappa-1)}$$

$$\geq \frac{\left[\underline{\Psi}(\kappa-1)^{\tau} \, \underline{\tilde{\theta}}(\kappa-1) \right]^2}{1 + \mu \, \underline{\Psi}(\kappa-1)^{\tau} \, \underline{\Psi}(\kappa-1)} \qquad (2.58)$$

where $\lambda\max[P(k-1)]$ is the maximum eigenvalue of $P(k-1)$ and μ is defined by

$$\mu = \frac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(0)^{-1} \right]}$$

Equation (2.54) and inequality (2.58) imply that

$$\lim_{\kappa\to\infty} \frac{\left[\underline{\Psi}(\kappa-1)^{\tau} \, \underline{\tilde{\theta}}(\kappa-1) \right]^2}{1 + \mu \, \underline{\Psi}(\kappa-1)^{\tau} \, \underline{\Psi}(\kappa-1)} = 0 \qquad (2.59)$$

which is equivalent to

$$\lim_{\kappa\to\infty} \frac{e(\kappa)^2}{1 + \mu \, \underline{\Psi}(\kappa-1)^{\tau} \, \underline{\Psi}(\kappa-1)} = 0 \qquad (2.60)$$

If the assumptions A2 and A3 are valid for any open loop system described by (2.24) then the regressor $\psi(k)$ will always remain bounded. This, by taking into account equation (2.60), means that

$$\lim_{\kappa\to\infty} e(\kappa) = 0$$

and property P3 is proven.

Next property P2 will be proven.

Equation (2.36) implies that

$$\underline{\tilde{\theta}}(\kappa) - \underline{\tilde{\theta}}(\kappa-1) = - \frac{\underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)\ \underline{\psi}(\kappa-1)^{\tau}\ \underline{\tilde{\theta}}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)}$$

Then

$$||\underline{\tilde{\theta}}(\kappa)-\underline{\tilde{\theta}}(\kappa-1)||^2 =$$

$$\left[- \frac{\underline{\underline{P}}(\kappa-1)\underline{\psi}(\kappa-1)\underline{\psi}(\kappa-1)^{\tau}\ \underline{\tilde{\theta}}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \right]^{\tau} \cdot \left[- \frac{\underline{\underline{P}}(\kappa-1)\underline{\psi}(\kappa-1)\underline{\psi}(\kappa-1)^{\tau}\ \underline{\tilde{\theta}}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \right]$$

$$= \left[\frac{\underline{\tilde{\theta}}(\kappa-1)^{\tau}\underline{\psi}(\kappa-1)\ \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \right] \left[\frac{\underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)\underline{\psi}(\kappa-1)^{\tau}\ \underline{\tilde{\theta}}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \right]$$

$$= \left[\frac{\underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)^2\ \underline{\psi}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \right] \left[\frac{\left[\underline{\psi}(\kappa-1)^{\tau}\ \underline{\tilde{\theta}}(\kappa-1)\right]^2}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \right] \qquad (2.61)$$

$$\frac{\underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)^2\ \underline{\psi}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)} \leq \bar{\lambda}_{\max}\left[\underline{\underline{P}}(\kappa-1)\right] \frac{\underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)}{1 + \underline{\psi}(\kappa-1)^{\tau}\ \underline{\underline{P}}(\kappa-1)\ \underline{\psi}(\kappa-1)}$$

$$\leq \bar{\lambda}_{\max}\left[\underline{\underline{P}}(\kappa-1)\right]$$

$$\leq \frac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa-1)^{-1}\right]}$$

$$\leq \left\{\prod_{i=1}^{\kappa-1} \bar{\lambda}(i)^{-1}\right\} \frac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(o)^{-1}\right]} \qquad (2.62)$$

Equality (2.61) and inequality (2.62) give

$$||\underline{\tilde{\theta}}(\kappa) - \underline{\tilde{\theta}}(\kappa-1)||^2 \leq \frac{1}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(o)^{-1}\right]} \{\prod_{i=1}^{\kappa-1} \lambda(i)^{-1}\} \frac{\left[\underline{\Psi}(\kappa-1)^\tau \, \underline{\tilde{\theta}}(\kappa-1)\right]^2}{1 + \underline{\Psi}(\kappa-1)^\tau \, \underline{\underline{P}}(\kappa-1) \, \underline{\Psi}(\kappa-1)}$$

The above inequality and the equation (2.54) imply that

$$\lim_{\kappa\to\infty} ||\underline{\tilde{\theta}}(\kappa) - \underline{\tilde{\theta}}(\kappa-1)||^2 = 0$$

which is equivalent to

$$\lim_{\kappa\to\infty} ||\underline{\tilde{\theta}}(\kappa) - \underline{\tilde{\theta}}(\kappa-1)\,|| = 0$$

The above equality proves property P2.

Finally, property P1 must be proven.

Equation (2.46) implies that

$$V(k) \leq \lambda(k)\, V(k-1) \tag{2.63}$$

From inequality (3.63), it is easily obtained that

$$V(\kappa) = \{\prod_{i=1}^{\kappa} \lambda(i)\} \;\; V(o)$$

which implies that

$$\underline{\tilde{\theta}}(\kappa)^\tau \, \underline{\underline{P}}(\kappa)^{-1} \, \underline{\tilde{\theta}}(\kappa) \leq \{\prod_{i=1}^{\kappa} \lambda(i)\} \;\; \underline{\tilde{\theta}}(o)^\tau \, \underline{\underline{P}}(o)^{-1} \, \underline{\tilde{\theta}}(o)$$

or

$$\bar{\lambda}_{\min}\left[\underline{\underline{P}}(\kappa)^{-1}\right] ||\underline{\tilde{\theta}}(\kappa)||^2 \leq \{\prod_{i=1}^{\kappa} \lambda(i)\} \;\; \bar{\lambda}_{\max}\left[\underline{\underline{P}}(o)^{-1}\right] ||\underline{\tilde{\theta}}(o)||^2$$

or

$$||\underline{\tilde{\theta}}(\kappa)||^2 \leq \frac{\bar{\lambda}_{\max}\left[\underline{\underline{P}}(o)^{-1}\right]}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(o)^{-1}\right]} \;\; ||\underline{\tilde{\theta}}(o)||^2$$

a or

$$||\underline{\tilde{\theta}}(\kappa)|| \leq \kappa_1 ||\underline{\tilde{\theta}}(o)|| \tag{2.64}$$

where

$$\kappa_1 = \frac{\bar{\lambda}_{\max}\left[\underline{\underline{P}}(o)^{-1}\right]}{\bar{\lambda}_{\min}\left[\underline{\underline{P}}(o)^{-1}\right]}^{\frac{1}{2}}$$

is the condition number of $P(0)^{-1}$. Usually, k_1 is chosen to be equal to one. For $k_1=1$, inequality (2.64) gives

$$||\underline{\tilde{\theta}}(\kappa)|| \leq ||\underline{\tilde{\theta}}(o)||$$

This inequality proves property P1. It also implies that the norm of the parameters estimates, given by this algorithm, will never be worse than that of the initial guess $\underline{\theta}(0)$ for the true parameter vector $\underline{\theta}_0$.

Despite its useful properties, the above analysed identification algorithm cannot guarantee that the parameters will converge to the true ones. In process identification the objective is to estimate the true parameter values in a finite time and not just have asymptotic convergence of the parameters (to any value that reduces the prediction error to zero). The next question is to see if there are any conditions under which the estimated parameters converge to the true ones. This is discussed in the next section.

2.4 Exponential Convergence

In a closed loop system we are primarily interested in keeping an output close to a desired value or to force an output to follow a desired trajectory. There are many cases where this objective can be achieved with only asympototically converged parameters (i.e. true parameter convergence is not a necessity). In any case, this does not mean that knowledge of the true parameters is not desirable. As it was previously mentioned, the idea of the certainty equivalence principle has been widely used in adaptive control. Obviously, fast identification of the true parameters will shorten the transient phase of the controller. The controller will converge to that one for which some stability results of the closed loop system have been derived. There are many cases (i.e. a pole assignment algorithm) where global stability of the closed loop system can be proven only if exponential convergence to the true parameters is guaranteed (Anderson and Johnson (1982), Anderson and Johnstone (1983)). Also systems which are exponentially convergent exhibit improved robustness properties (Johnstone (1982)). For the simple identification problem, exponential convergence to the true parameters can be defined as the desired objective.

The identification scheme, analysed in the previous section, cannot guarantee that exponential convergence will occur. It is our objective to seek conditions under which the identification scheme used also guarantees exponential convergence.

Identification of a process cannot be achieved unless the input signal excites all the modes of the system. So, the input signal must have sufficient energy to do it. In other words it must be persistently exciting. But how is a persistently exciting signal defined? Roughly speaking, a signal is persistently exciting if it is different than zero. The next question is: are all the nonzero signals persistently exciting? The answer is negative. For example, let us consider the constant unit signal i.e. $u(k)=1$ for all k. Then equations (2.1), (2.2) and (2.3) give that

$$y(k)=a_1\, y(k-1) + \ldots + a_n\, y(k-n) + b_{d+1} + \ldots\, b_{d+m}$$

It is very easily seen from the above equation that we cannot identify each of the b's parameters but only the sum of them (Aström (1966)).

The next step is to try to define a persistently exciting signal. Consider the equation (2.47). This equation, for $\lambda(k)<1$, gives that $V_{\infty}=0$ So, for $\lambda(k)$ less than one, the Lyaponov type scalar function $V(k)$ converges to zero. By definition

$$V(k) = \tilde{\underline{\theta}}(k)^{T}\, \underline{\underline{P}}(k)^{-1}\, \tilde{\underline{\theta}}(k)$$

Since our objective is to get an estimate of the true process parameters, at the limit we want $V(k)$ to approach to zero not because $\underline{\underline{P}}(k)^{-1}$ approaches to zero but because $\underline{\theta}(k)$ becomes zero. From equation (3.39) it can be shown that

$$\underline{\underline{P}}(k)^{-1} = \lambda\,(1)\, \underline{\underline{P}}(0)^{-1} + \sum_{i=1}^{k} \lambda\,(i)\, \underline{\Psi}(k-i)\, \underline{\Psi}(k-i)^{T} \qquad (2.66)$$

The matrix $P(k)^{-1}$ will be kept away from zero if

$$\rho_1 \leq \bar{\lambda}_{\min} \left[\sum_{i=1}^{\kappa} \underline{\psi}(\kappa-i)\, \underline{\psi}(\kappa-i)^{\tau} \right] \tag{2.67}$$

where ρ_1 is a suitable positive number. On the other hand we want to identify the correct parameters by using bounded signals only. This means that the regressor $\underline{\psi}(k)$ must be bounded. Under this requirement, inequality (2.67) gives

$$\rho_1 \leq \bar{\lambda}_{\min} \left[\sum_{i=1}^{\kappa} \underline{\psi}(\kappa-i)\, \underline{\psi}(\kappa-i)^{\tau} \right] \leq \rho_2 \tag{2.68}$$

where ρ_2 is a suitable positive number. The previous analysis is quite empirical but it gives an insight into the definition of a persistently exciting signal found in the literature.

According to Anderson and Johnson (1982) a signal u(k) is said to be persistently exciting if

$$0 < \rho_3 \underline{\underline{I}} < \sum_{J}^{J+S} \begin{bmatrix} u(\kappa) \\ u(\kappa-1) \\ \vdots \\ u(\kappa-\rho) \end{bmatrix} \begin{bmatrix} u(\kappa) & u(\kappa-1) & \ldots u(\kappa-\rho) \end{bmatrix} < \rho_4 \underline{\underline{I}} \tag{2.69}$$

for some ρ_3, ρ_4, ρ, s and all j. With s, we denote the time interval over which the signal is persistently exciting. Since inequalities between matrices are not allowed, inequality (2.69) is valid only in terms of minimum eigenvalues.

In the sequel, we shall prove that the identification scheme (2.24) to (2.30) is exponentially convergent if the regressor $\underline{\psi}(k)$ is persistently exciting. A similar result has also been derived by Johnstone (1982). Our reslt, presented by Theorem 2.3, can be considered as an extension of Johnstone's work for the case that a variable forgetting factor is used. Before we prove Theorem 2.3, the covariance matrix must be proven to be bounded from above and below. This is done in the following theorem.

Theorem 2.2

If the regressor is persistently exciting in the sense that for some constant integer s and all j there exists positive constants ρ_1 and ρ_2 such that:

$$0 \le \rho_1 \underline{\underline{I}} \le \sum_{\kappa=J}^{J+S} \underline{\Psi}(\kappa-1)\, \underline{\Psi}(\kappa-1)^{\tau} \le \rho_2 I < \infty \tag{2.70}$$

then, for all $k \ge s+1$

$$0 < \frac{\rho_1 \left[\dfrac{1}{\lambda_{min}} - 1\right]}{\left(\dfrac{1}{\lambda_{min}}\right)\left[\left(\dfrac{1}{\lambda_{min}}\right)^{S+1} - 1\right]} \underline{\underline{I}} \le \underline{\underline{P}}(\kappa)^{-1} \le \lambda_{max}^{\kappa-1} \sum_{i=0}^{S} \underline{\underline{P}}(i)^{-1} + \frac{1}{1-\lambda_{max}} a_2 \underline{\underline{I}}$$

where λmin and λmax are the minimum and maximum allowable values for the forgetting factor, respectively.
More specifically: $0<\lambda min \le \lambda(k) \le \lambda max<1$.

Proof:

From equation (2.39) we can conclude that

$$\lambda(\kappa)\, \underline{\Psi}(\kappa-1)\, \underline{\Psi}(\kappa-1)^{\tau} \le \underline{\underline{P}}(\kappa)^{-1} \tag{2.71}$$

The comparison of the matrices, in inequality (2.71), is in terms of minimum eigenvalues. It simply means that the matrix $\underline{P}(k)^{-1}$ is more positive than the matrix $\lambda(k)\ \underline{\psi}(k-1)\ \underline{\psi}(k-1)'$. Inequality (2.71) also means that

$$\sum_{\kappa=j}^{j+s} \underline{\Psi}(\kappa-1)\ \underline{\Psi}(\kappa-1)^{\tau} < \sum_{\kappa=j}^{j+s} \frac{1}{\lambda(\kappa)}\ \underline{\underline{P}}(\kappa)^{-1} \tag{2.72}$$

Inequalities (2.70) and (2.72) give that

$$o < \rho_1\ \underline{\underline{I}} \leq \sum_{\kappa=j}^{j+s} \frac{1}{\lambda(\kappa)}\ \underline{\underline{P}}(\kappa)^{-1} \tag{2.73}$$

or equivalently

$$\rho_1\ \underline{\underline{I}} \leq \left[\frac{1}{\lambda(j)}\ \underline{\underline{P}}(j)^{-1} + \frac{1}{\lambda(j+1)}\ \underline{\underline{P}}(j+1)^{-1} + \ldots \frac{1}{\lambda(j+s)}\ \underline{\underline{P}}(j+s)^{-1} \right] \tag{2.74}$$

Also from equation (2.39)

$$\lambda(\kappa)\ \underline{\underline{P}}(\kappa-1)^{-1} \leq \underline{\underline{P}}(\kappa)^{-1} \tag{2.75}$$

By using (2.75), inequality (2.74) can be written as

$$a_1\ \underline{\underline{I}} \leq \left[\frac{1}{\prod_{\kappa=j}^{j+s} \lambda(\kappa)} + \frac{1}{\prod_{\kappa=j+1}^{j+s} \lambda(\kappa)} + \ldots + \frac{1}{\prod_{\kappa=j+s}^{j+s} \lambda(\kappa)} \right] \underline{\underline{P}}(j+s)^{-1} \tag{2.76}$$

At this point we introduce a lower bound, λmin, for the variable forgetting factor λ(k) such that

$$\lambda\text{min} \leq \lambda(k) \text{ for all } k \tag{2.77}$$

Inequalities (2.76) and (2.77) give

$$\rho_1\ \underline{\underline{I}} \leq \left[\frac{1}{\lambda_{\text{min}}^{s+1}} + \frac{1}{\lambda_{\text{min}}^{s}} + \ldots + \frac{1}{\lambda_{\text{min}}} \right] \underline{\underline{P}}(j+s)^{-1} \tag{2.78}$$

If S_{j+s} stands for the summation in brackets in inequality (2.78), the following will be valid

$$S_{j+s} = \frac{\frac{1}{\lambda_{min}} \cdot \left[\left[\frac{1}{\lambda_{min}}\right]^{s+1} - 1\right]}{\left[\frac{1}{\lambda_{min}} - 1\right]}$$

Then

$$0 < \frac{\rho_1 \left[\frac{1}{\lambda_{min}} - 1\right]}{\frac{1}{\lambda_{min}} \cdot \left[\left[\frac{1}{\lambda_{min}}\right]^{s+1} - 1\right]} \underline{\underline{I}} \le \underline{\underline{P}}(j+s)^{-1}$$

or

$$0 < \frac{\rho_1 \left[\frac{1}{\lambda_{min}} - 1\right]}{\frac{1}{\lambda_{min}} \cdot \left[\left[\frac{1}{\lambda_{min}}\right]^{s+1} - 1\right]} \underline{\underline{I}} \le \underline{\underline{P}}(\kappa)^{-1} \qquad (2.79)$$

for all $k \geq s+1$ and this completes the proof.

In order to prove the upper bound of $\underline{P}(k)^{-1}$, equation (2.39) is used. From equation (2.39)

$$\underline{\underline{P}}(\kappa)^{-1} \le \lambda(\kappa)\ \underline{\underline{P}}(\kappa-1)^{-1} + \underline{\Psi}(\kappa-1)\ \underline{\Psi}(\kappa-1)^{\tau}$$

or

$$\sum_{\kappa=j}^{j+s} \underline{\underline{P}}(\kappa)^{-1} \le \sum_{\kappa=j}^{j+s} \lambda(\kappa)\ \underline{\underline{P}}(\kappa-1)^{-1} + \sum_{\kappa=j}^{j+s} \underline{\Psi}(\kappa-1)\ \underline{\Psi}(\kappa-1)^{\tau}$$

and by using (2.70)

$$\sum_{\kappa=j}^{j+s} \underline{\underline{P}}(\kappa)^{-1} \le \sum_{\kappa=j}^{j+s} \lambda(\kappa)\ \underline{\underline{P}}(\kappa-1)^{-1} + \alpha_2\ \underline{\underline{I}} \tag{2.80}$$

If we impose an upper bound λmax for the variable forgetting factor λ(k), inequality (2.80) will give

$$\underline{\underline{P}}(j)^{-1} + \underline{\underline{P}}(j+1)^{-1} + \ldots + \underline{\underline{P}}(j+s)^{-1} \le$$

$$\lambda_{max} \cdot \left[\underline{\underline{P}}(j-1)^{-1} + \underline{\underline{P}}(j)^{-1} + \ldots\ \underline{\underline{P}}(j+s-1)^{-1} \right] + \alpha_2\ \underline{\underline{I}} \qquad (2.81$$

Inequality (2.81) gives

$$\begin{array}{lcl}
\underline{\underline{P}}(j-1)^{-1} + \ldots + \underline{\underline{P}}(j+s-1)^{-1} & \le & \lambda_{max} \cdot \left[\underline{\underline{P}}(j-2)^{-1} + \ldots + \underline{\underline{P}}(j+s-2)^{-1} \right] + \alpha_2\ \underline{\underline{I}} \\
\vdots & \vdots & \vdots \\
\underline{\underline{P}}(1)^{-1} + \ldots + \underline{\underline{P}}(s+1)^{-1} & \le & \lambda_{max} \cdot \left[\underline{\underline{P}}(o)^{-1} + \ldots + \underline{\underline{P}}(s)^{-1} \right] + \alpha_2\ \underline{\underline{I}}
\end{array}$$

From inequality (2.80) and the set of the above inequalities, the following results

$$\sum_{\kappa=j}^{j+s} \underline{\underline{P}}(\kappa)^{-1} \le \lambda_{\max}^{\,j-1} \sum_{i=0}^{s} \underline{\underline{P}}(i)^{-1} + \left[1 + \lambda_{\max} + \dots + \lambda_{\max}^{\,j-1} \right] \alpha_2 \underline{\underline{I}}$$

or

$$\underline{\underline{P}}(j)^{-1} \le \lambda_{\max}^{\,j-1} \sum_{i=0}^{s} \underline{\underline{P}}(i)^{-1} + \frac{1 - \lambda_{\max}^{\,j}}{1 - \lambda_{\max}} \alpha_2 \underline{\underline{I}}$$

or

$$\underline{\underline{P}}(\kappa)^{-1} \le \lambda_{\max}^{\,\kappa-1} \sum_{i=1}^{s} \underline{\underline{P}}(i)^{-1} + \frac{1}{1 - \lambda_{\max}} \alpha_2 \underline{\underline{I}} \qquad (2.81)$$

and this completes the proof.

Remark 2.1

It is very interesting to see from inequalities (2.79) and (2.81) that the inverse of the covariance matrix $\underline{\underline{P}}(k)$ will always remain bounded away from zero and infinity if the regressor is persistently exciting. This automatically means that the covariance matrix will remain bounded too. So, by using persistently exciting signals, the "drifting" and "blow up" problems are eliminated.

Theorem 2.3

If the sequence $\underline{\psi}(k-1)$ is persistently exciting then the recursive least squares algorithm (2.24) to (2.30) with variable forgetting factor $(\lambda_{min} \le \lambda(k) \le \lambda_{max} < 1)$ is exponentially stable.

Proof:

By definition

$$V(\kappa) = \underline{\tilde{\theta}}(\kappa)^T \, \underline{\underline{P}}(\kappa)^{-1} \, \underline{\tilde{\theta}}(\kappa)$$

Recall that

$$V(\kappa) \le \lambda(\kappa) \; V(\kappa-1)$$

Then,

$$V(\kappa) \le \{\prod_{i=1}^{\kappa} \lambda(i)\} \, V(0)$$

or

$$\underline{\tilde{\theta}}(\kappa) \; \underline{\underline{P}}(\kappa)^{-1} \; \underline{\tilde{\theta}}(\kappa) \le \{\prod_{i=1}^{\kappa} \lambda(i)\} \; \underline{\tilde{\theta}}(0) \; \underline{\underline{P}}(0)^{-1} \; \underline{\tilde{\theta}}(0)$$

or

$$\bar{\lambda}_{min}\left[\underline{\underline{P}}(\kappa)^{-1}\right] ||\tilde{\theta}(\kappa)||^2 \le \{\prod_{i=1}^{\kappa} \lambda(i)\} \; \bar{\lambda}_{max}\left[\underline{\underline{P}}(0)^{-1}\right] ||\tilde{\theta}(0)||^2$$

and by using (2.79)

$$\frac{\rho_1 \left[\frac{1}{\lambda_{min}} - 1\right]}{\frac{1}{\lambda_{min}} \cdot \left[\left[\frac{1}{\lambda_{min}}\right]^{s+1} - 1\right]} ||\underline{\tilde{\theta}}(\kappa)||^2 \le \{\prod_{i=1}^{\kappa} \lambda(i)\} \; \bar{\lambda}_{max}\left[\underline{\underline{P}}(0)^{-1}\right] ||\underline{\tilde{\theta}}(0)||^2$$

or

$$||\underline{\tilde{\theta}}(\kappa)||^2 \le \frac{\frac{1}{\lambda_{min}}\left[\left[\frac{1}{\lambda_{min}}\right]^{s+1} - 1\right]}{\rho_1\left[\frac{1}{\lambda_{min}} - 1\right]} \{\prod_{i=1}^{\kappa} \lambda(i)\} \; \bar{\lambda}_{max}\left[\underline{\underline{P}}(0)^{-1}\right] ||\underline{\tilde{\theta}}(0)||^2 \tag{2.82}$$

Inequality (2.81) proves the exponential stability of the identification scheme under the assumption that an upper bound λmax, less than one, for the variable forgetting factor is used.

It is interesting to see that the conditions, necessary for exponential convergence of the identification scheme, are exactly the same as those mentioned in the discussion when we tried to give an insight into the definition of a persistently exciting signal. It also must be mentioned that, in an identification problem, the only signal that we can manipulate is the process input. We can force the input to be persistently exciting but nothing can be said about the regressor. It was not until 1982 when Anderson and Johnson (1982) proved that for a DARMA model a persistently exciting input guarantees that the regressor will be persistently exciting.

Having seen the usefulness of persistently exciting signals, the question to pose is: Is there any particular class of persistently exciting signals? One type of signals, satisfying this requirement, is the periodic signals (Lion (1967)). Also, Yuan and Wonham (1977) discuss the design of input signals (or probing signals) to ensure exponential convergence. Most of the signals, that they mention in their work, are periodic. Exponential convergence is always desirable but in a closed-loop system the excitation of the process must be minimal. At least for the continuous time domain, there are theoretical results (Boyd and Sastry

(1983)) which relate the amount of minimum excitation to the number of parameters that must be identified. But nothing is known about the required energy of the persistently exciting signals.

Having analysed the recursive least squares identificaton algorithm (2.24) to (2.30), some simulation results are presented, in the next section, to illustrate its performance.

2.5 Simulation Results

In this section, we shall examine the performance of the identification algorithm (2.24) to (2.30) by identifying a second order process. Also, the significance of different parameters, which are user specified, will be considered.

We consider the following process:

$$y(k)=1.2\ y(k-1) - 0.52\ y(k-2) + 0.2\ u(k-1) + 0.12\ u(k-2)$$

This process is identified under different conditions, as shown in Table 2.1. In figure 2.1, the process input changes from 1 to -1 every five sampling intervals. By applying this type of input, the process output responds with a trajectory which is in figure 2.2. From figure 2.2 it can be seen that the process and model outputs match very well from the very beginning. This is due to the exact estimation of the true process parameters, as it can be seen from figures 2.3 and 2.4. The trajectory of the forgetting factor can be seen in figure 2.5. After the first iteration, the forgetting factor drops abruptly because of the prediction error. The forgetting factor goes to its upper limit, when the prediction error becomes zero. By comparing figure 2.3 with figure 2.7 and figure 2.4 with figure 2.8, we can see the effect of the parameter N_0, when all the other parameters are kept constant. We can see that a large variation of the N_0 does not cause any significant change in the estimation of the parameters. We cannot generalize it since, in our case, the prediction error becomes zero very fast.

Next we want to illustrate the effect of the initial value of the covariance matrix on the identification performance. Figures 2.9 and 2.10 have been drawn under the same conditions as the figures 2.3 and 2.4 have except that the initial covariance matrix is 50 $\underline{I}$ instead of 10^4 $\underline{I}$. The covariance matrix reflects our confidence about the initial parameter vector $\underline{\theta}(0)$. A small initial value means that our guess for $\underline{\theta}(0)$ is good. So, the estimation must proceed slowly as it appears to do in figures 2.9 and 2.10. In figures 2.3 and 2.4 the large initial value for the covariance matrix leads to sharp changes for the parameter estimates between successive sampling instants.

Figures 2.15 and 2.16 have been drawn for a variable forgetting factor less than one (λmax=0.9) The process input used in this simulation run is of the input form shown in figure 2.1. Despite the large number of iterations (1500 in this case), the "blow up" problem has been avoided because the excitation introduced by the input signal assures the boundness of the covariance matrix. This agrees with inequalities (2.79) and (2.81) which have been derived under the assumption that the regressor is persistently exciting. The same simulation run has been repeated with a constant input signal equal to one. The "blow up" problem occured after 249 iterations because of the insufficient excitation.

Figures 2.13 to 2.16 have been drawn under the same conditions as figures 2.17 to 2.20. From these figures we can conclude that if we do not excite the process quite frequently this can lead to the "blow up" problem. By applying a stepwise input which changes from 1 to -1 every 200 sampling intervals, the "blow up" problem occurs after 131 iterations (figure 2.13). This problem can be avoided if we apply a more exciting (or dynamic) input (figure 2.17). So it is not always useful to use persistently exciting signals i.e. one could only have a stationary step-input and yet not have convergence. For exponential convergence one needs persistently exciting signals of sufficient energy that are "exciting" in the magnitude as well as the frequency content. Once convergence has taken place then a practical approach to prevent the 'blow-up' problem in the absence of persistent excitation is to change λmax=1.0.

TABLE 2.1

Summary of the simulation results illustrating the performance of the identification scheme.

Simulation Run	N_0	$\underline{P}(0)$	λmax	λmin	Figure Number
1	10	10^6	0.98	0.9	(2.1) to (2.6)
2	1000	10^6	0.98	0.9	(2.7) to (2.8)
3	10	50	0.95	0.9	(2.9) to (2.10)
4	10	10^6	0.95	0.9	(2.11) to (2.12)
5	10	10^6	0.90	0.7	(2.13) to (2.16)
6	10	10^6	0.90	0.7	(2.17) to (2.20)

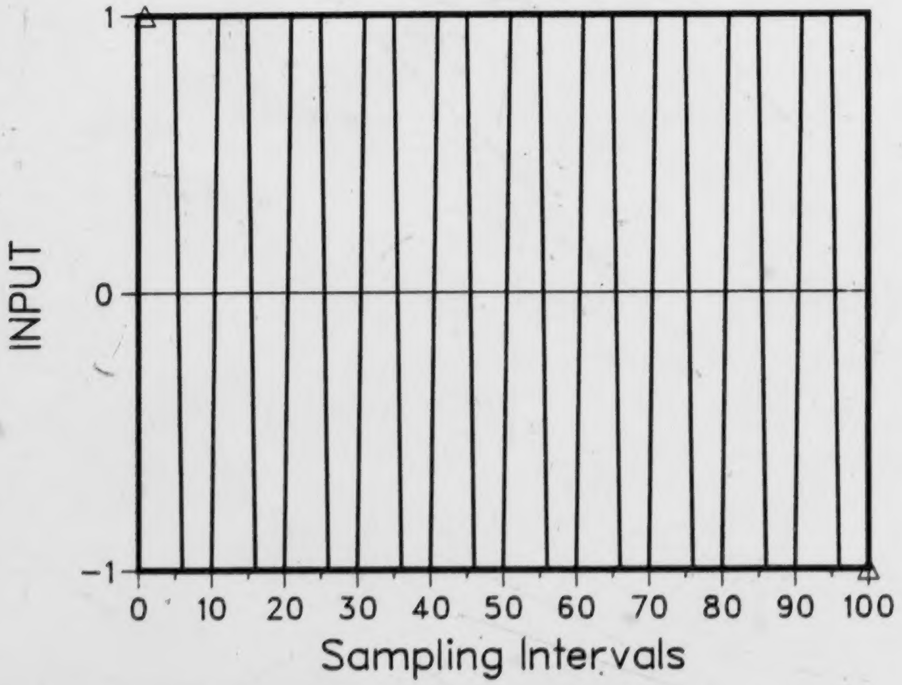


Fig.−2.1 Process input trajectory

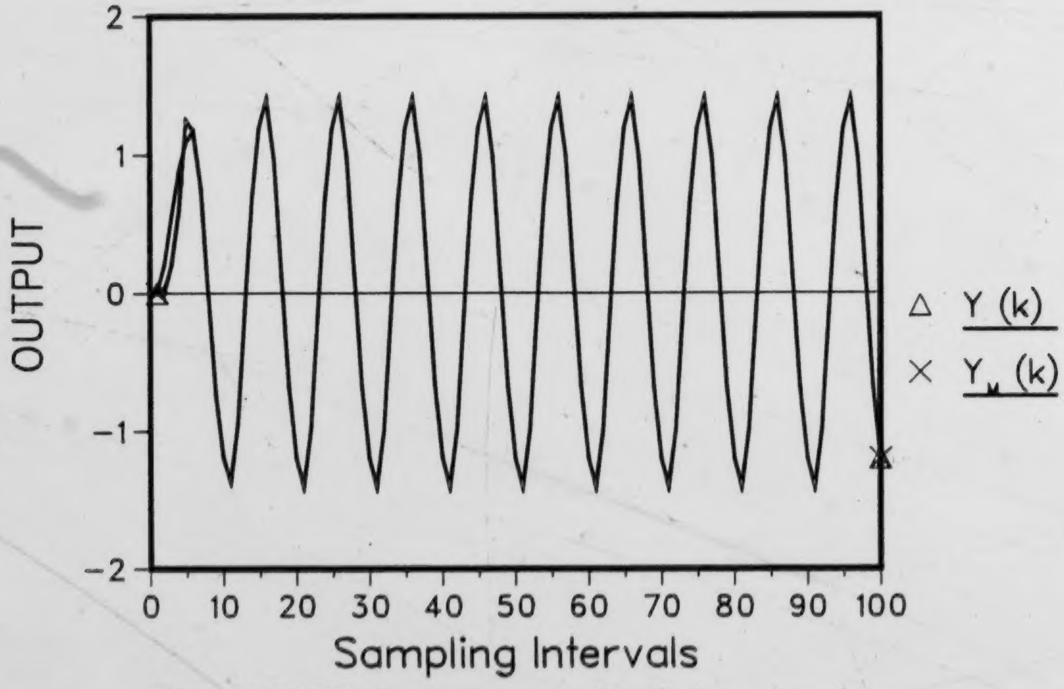


Fig.−2.2 Process and Model output trajectories

$N_o=10$ $P(0)=1.E6$ $\lambda_{max}=0.98$

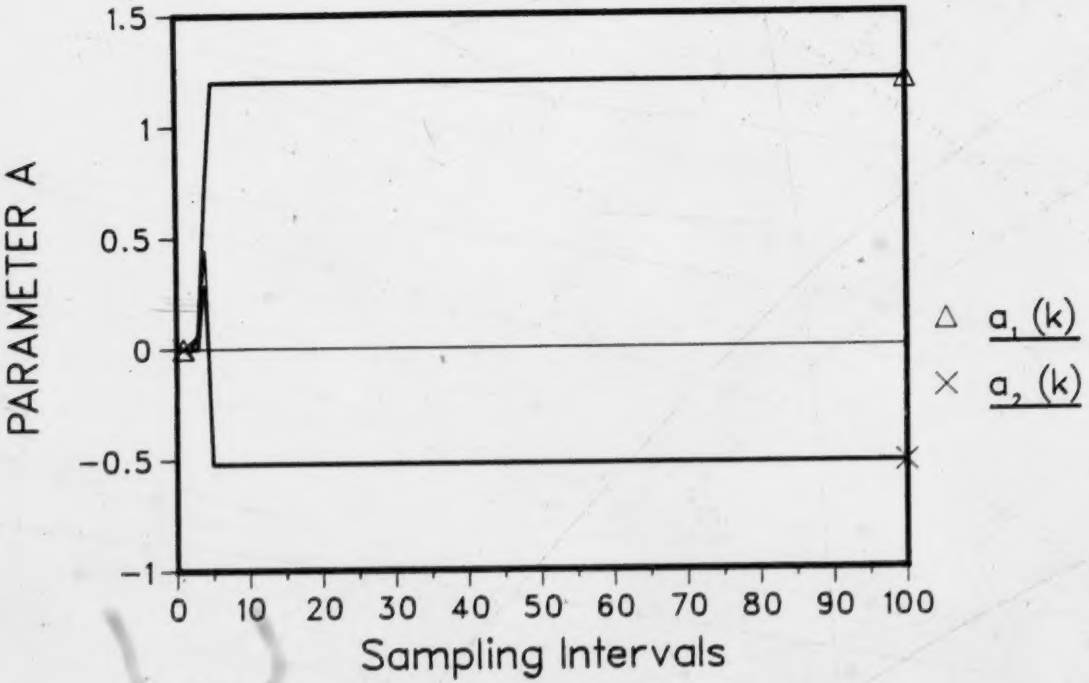


Fig.–2.3 Process parameter trajectory

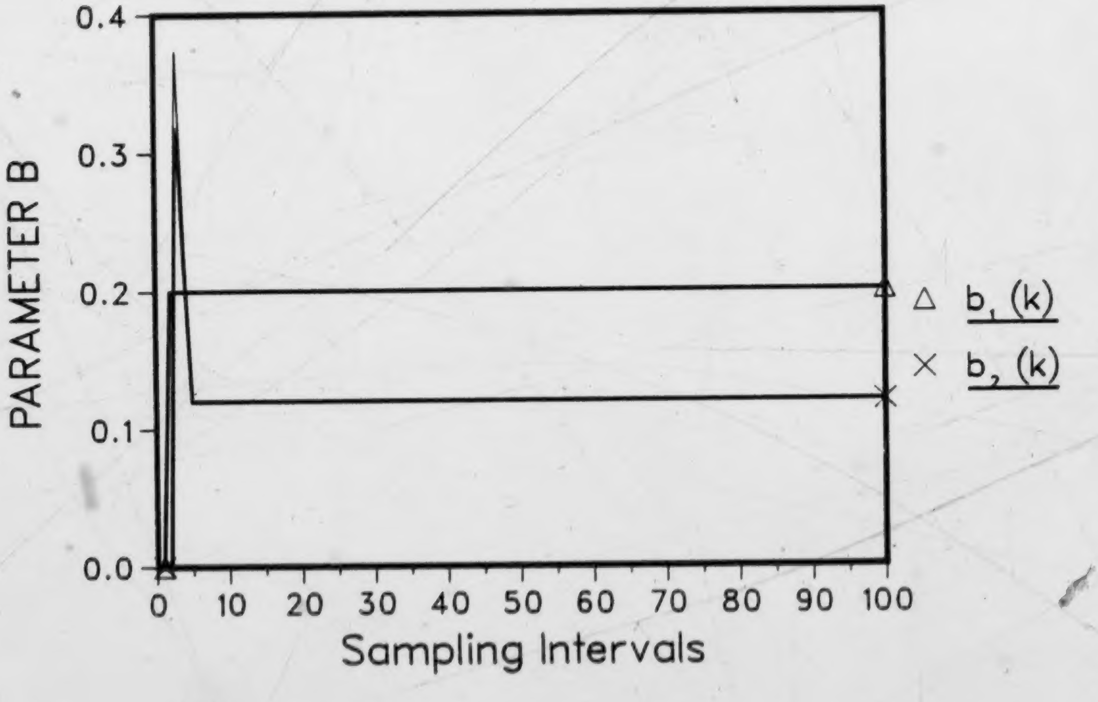


Fig.–2.4 Process parameter trajectory
N_o=10 P(0)=1.E6 λ_{max}=0.98

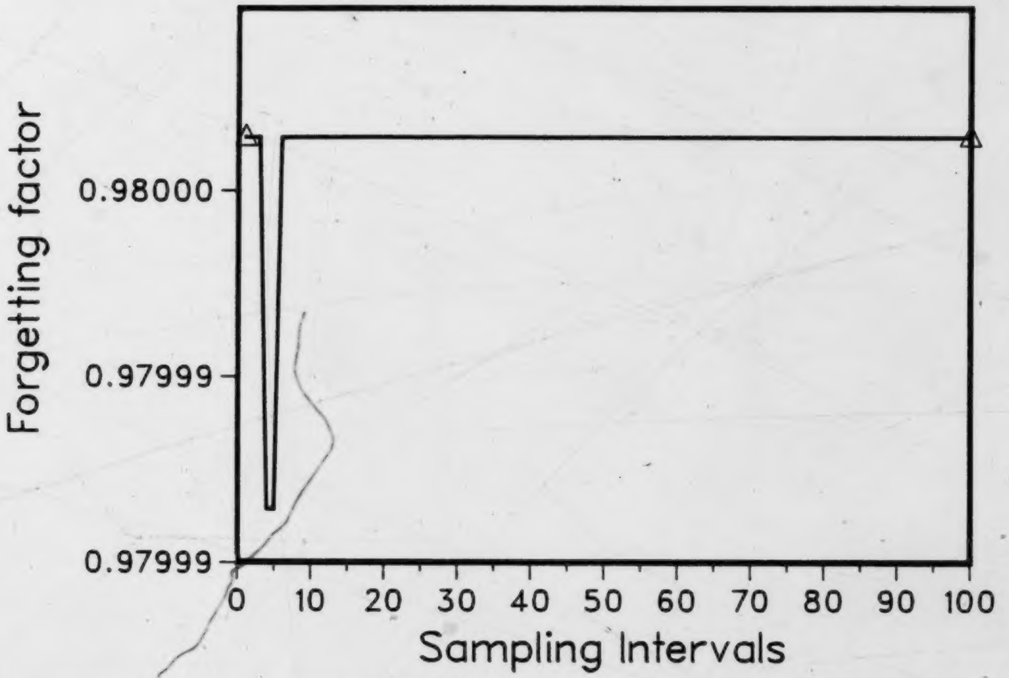


Fig.—2.5 Forgetting factor trajectory

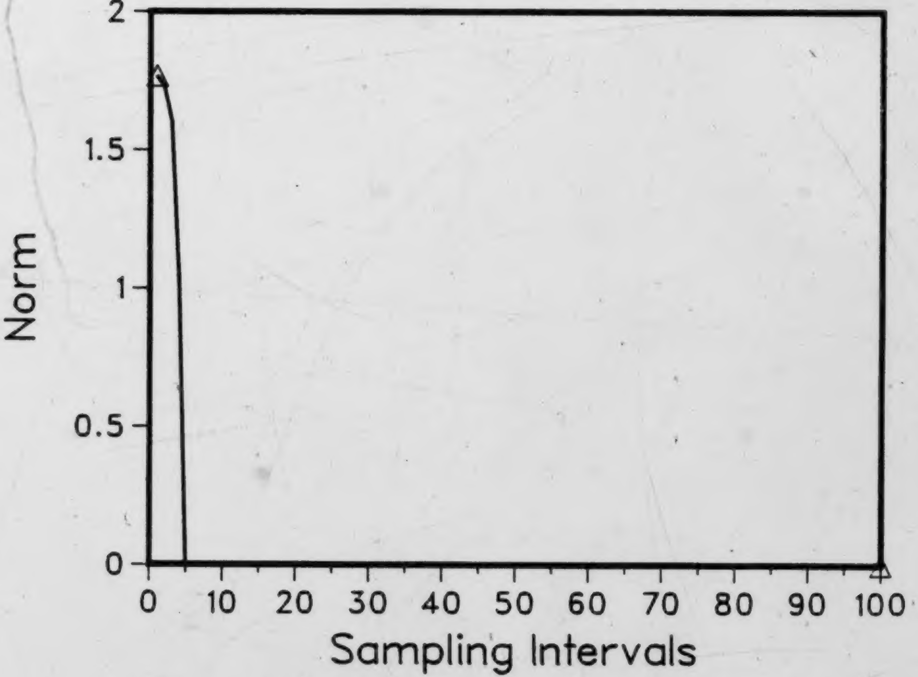


Fig.—2.6 Parameter error vector norm trajectory

N_o=10 P(0)=1.E6 λ_{max}=0.98

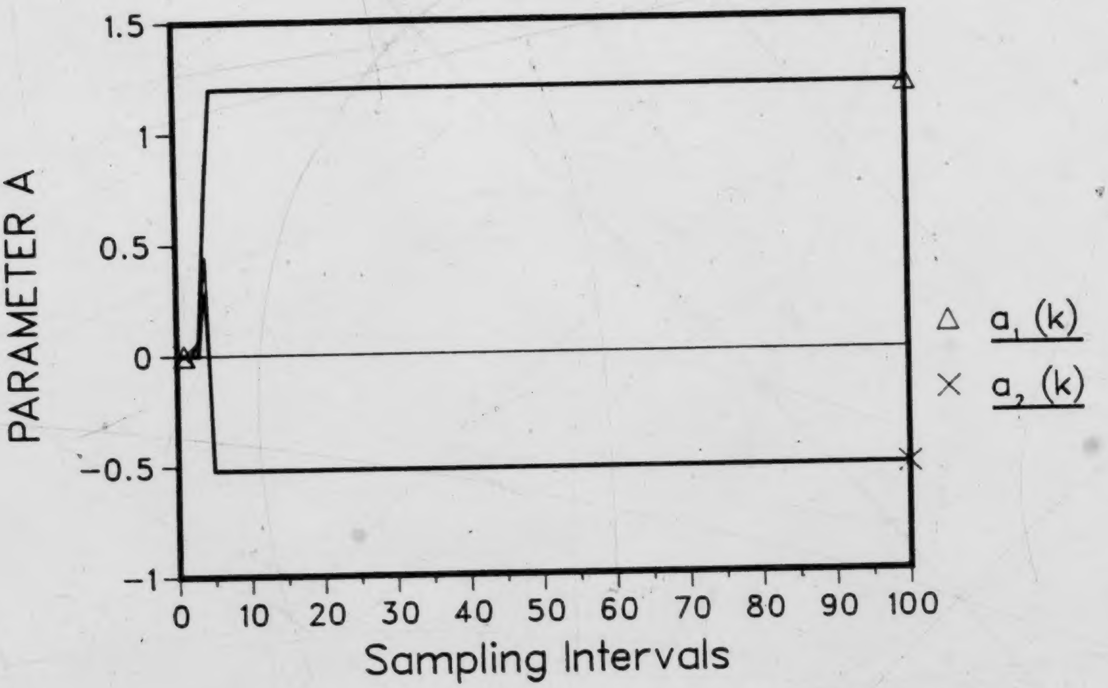


Fig.–2.7 Process parameter trajectory

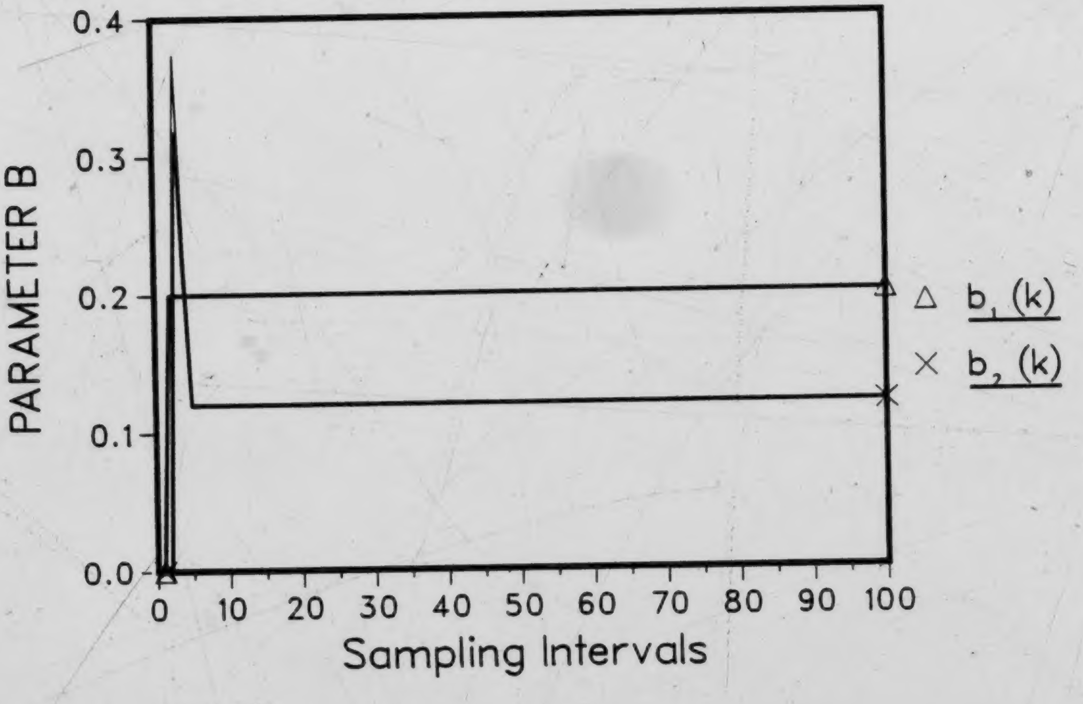


Fig.–2.8 Process parameter trajectory
N_0=1000 P(0)=1.E6 λ_{max}=0.98

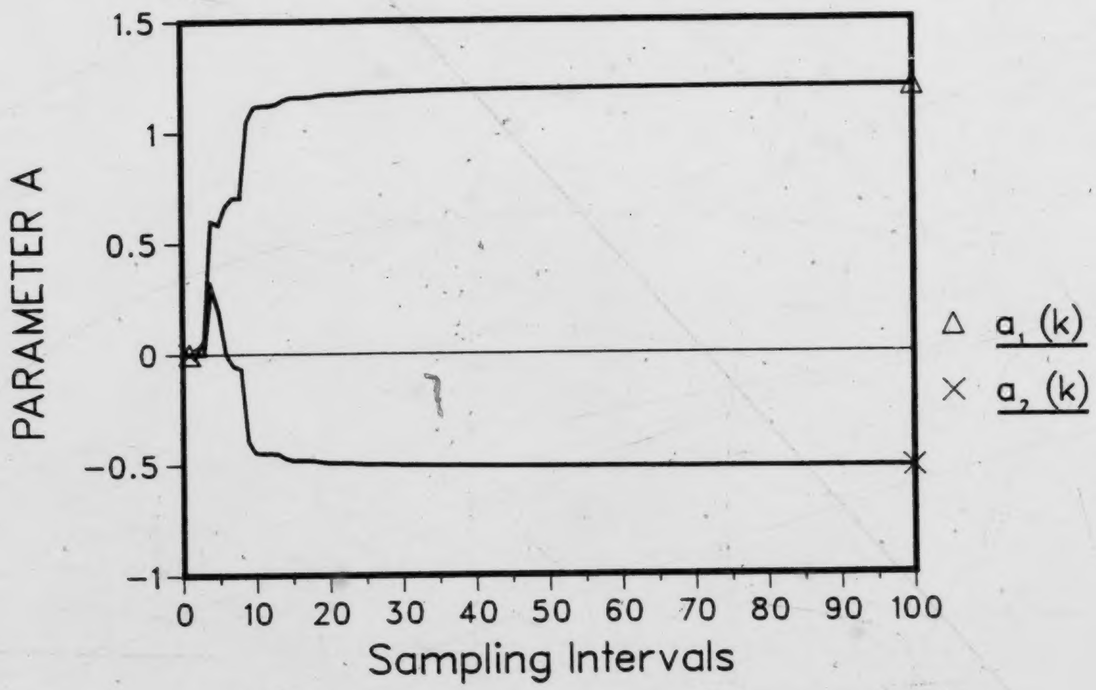


Fig.−2.9 Process parameter trajectory

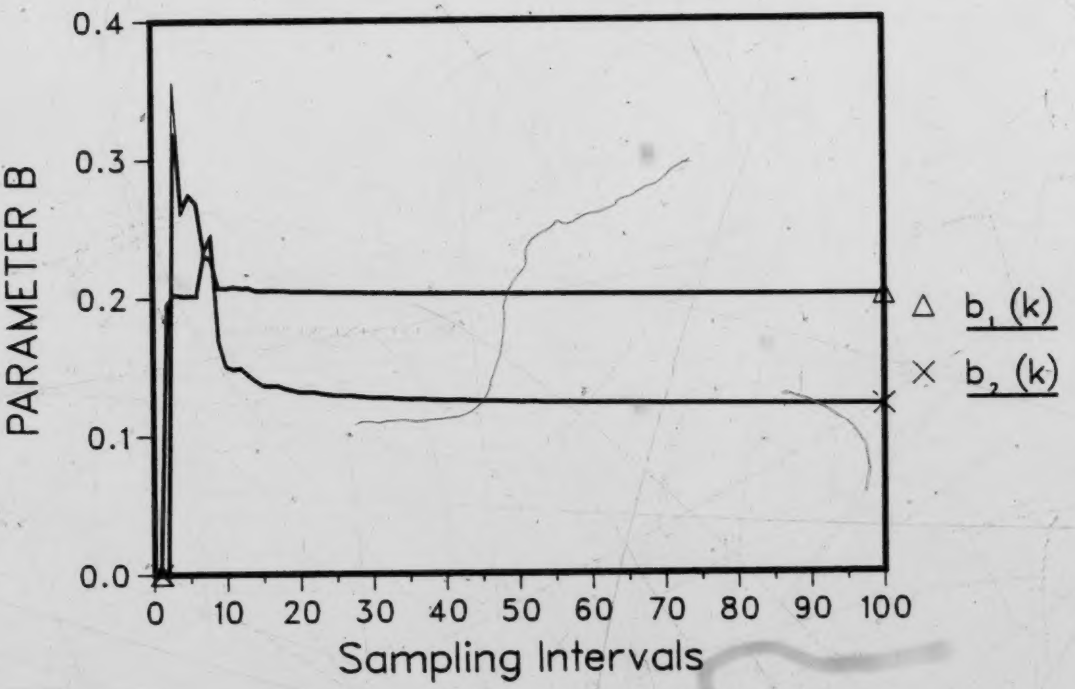


Fig.−2.10 Process parameter trajectory
N_0=10 P(0)=50 λ_{max}=0.98

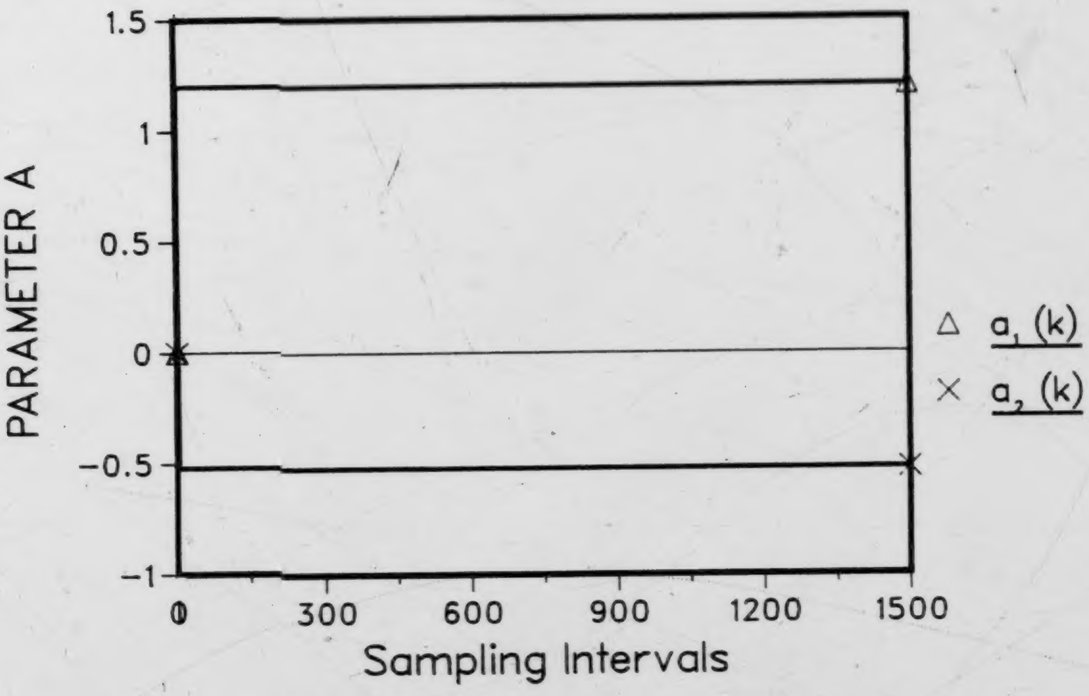


Fig.-2.11 Process parameter trajectory

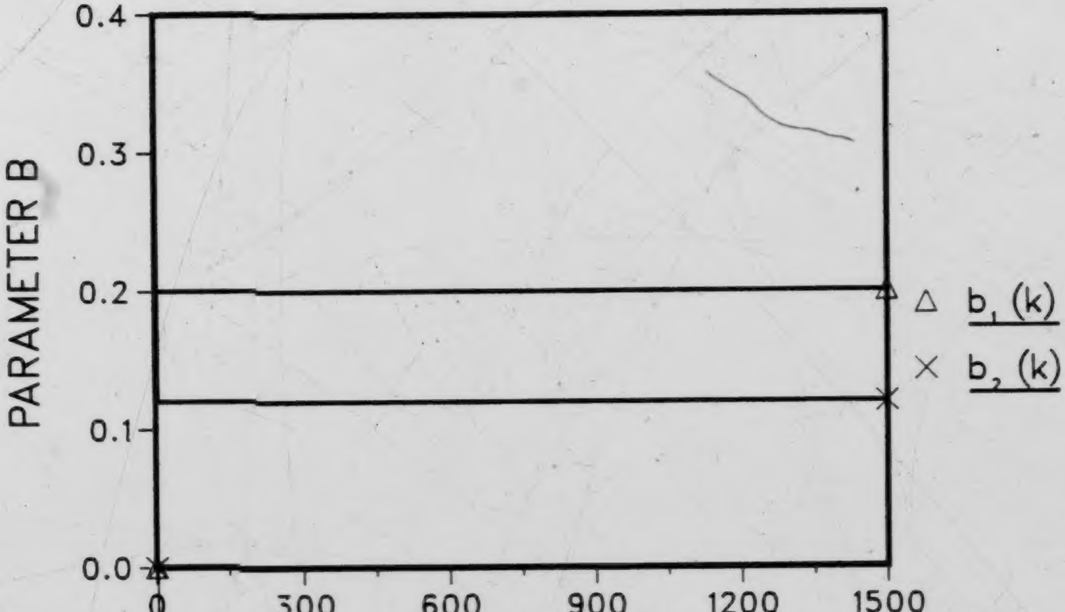


Fig.-2.12 Process parameter trajectory
N_o=10 P(0)=1.E6 λ_{max}=0.95

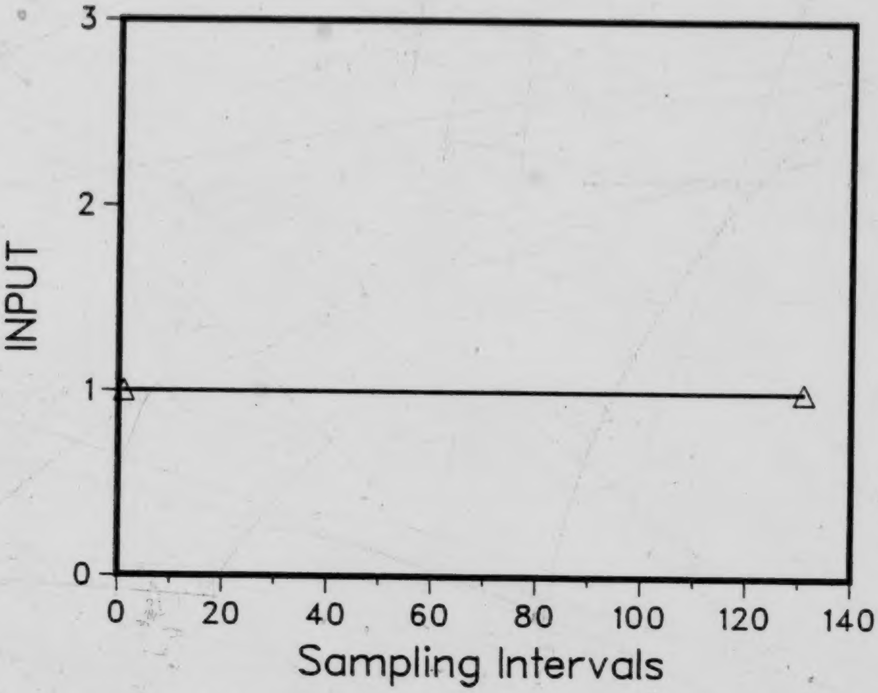


Fig.–2.13 Process input trajectory

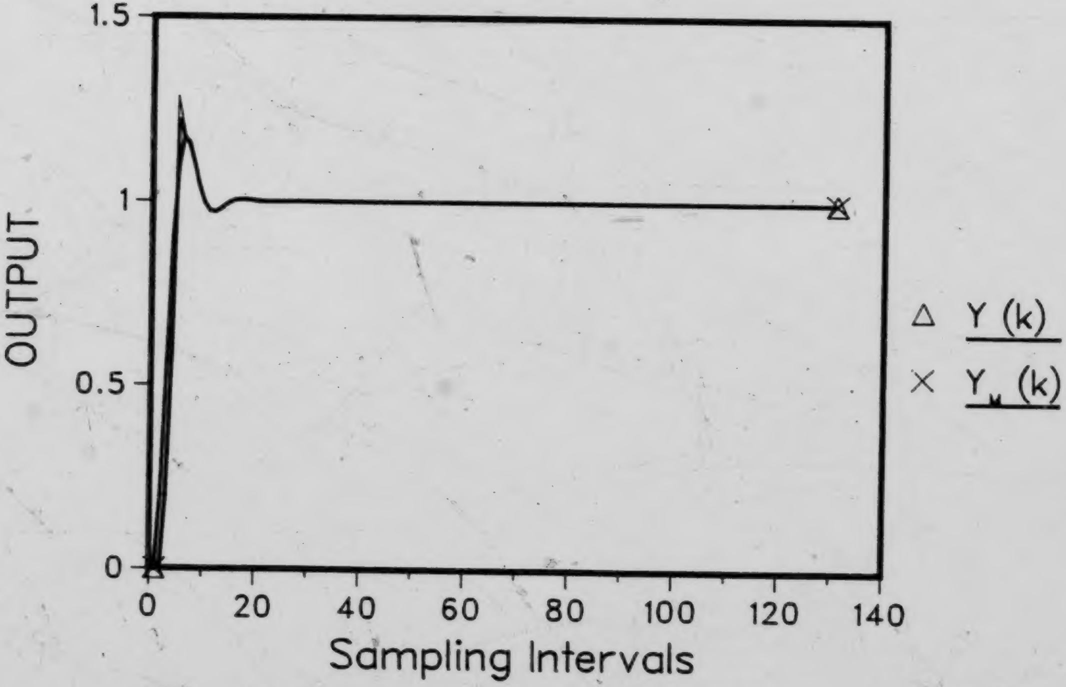


Fig.–2.14 Process and Model output trajectories

$N_o=10$ $P(0)=1.E6$ $\lambda_{max}=0.9$

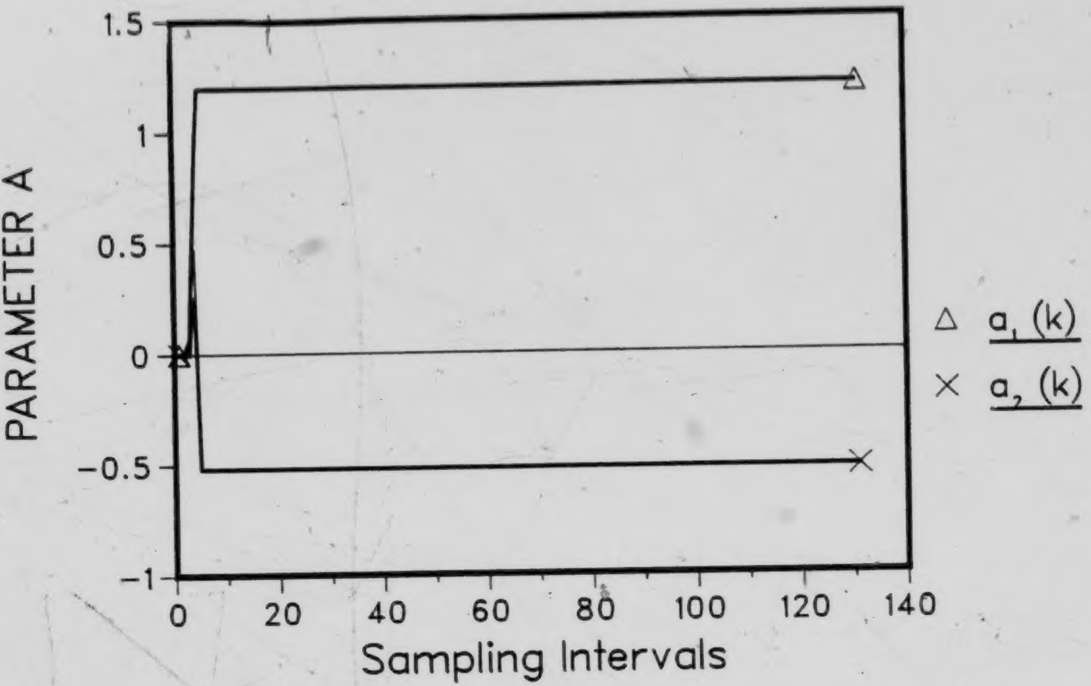


Fig.–2.15 Process parameter trajectory

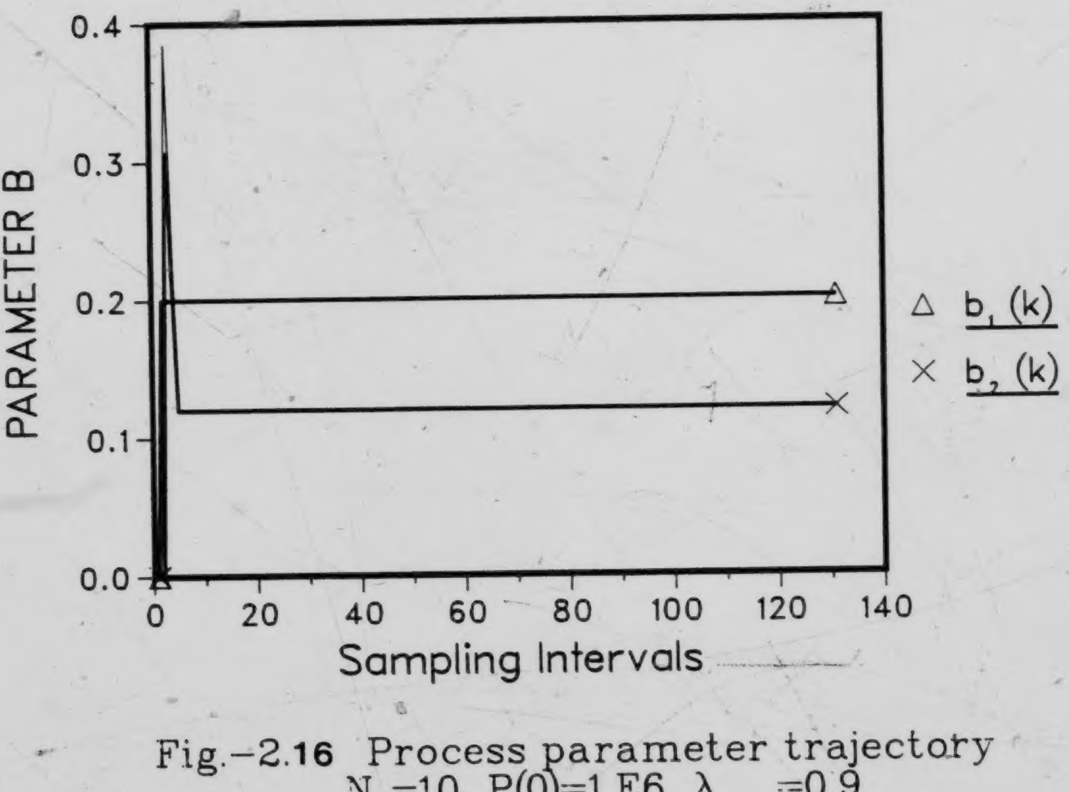


Fig.–2.16 Process parameter trajectory
N_0=10 P(0)=1.E6 λ_{max}=0.9

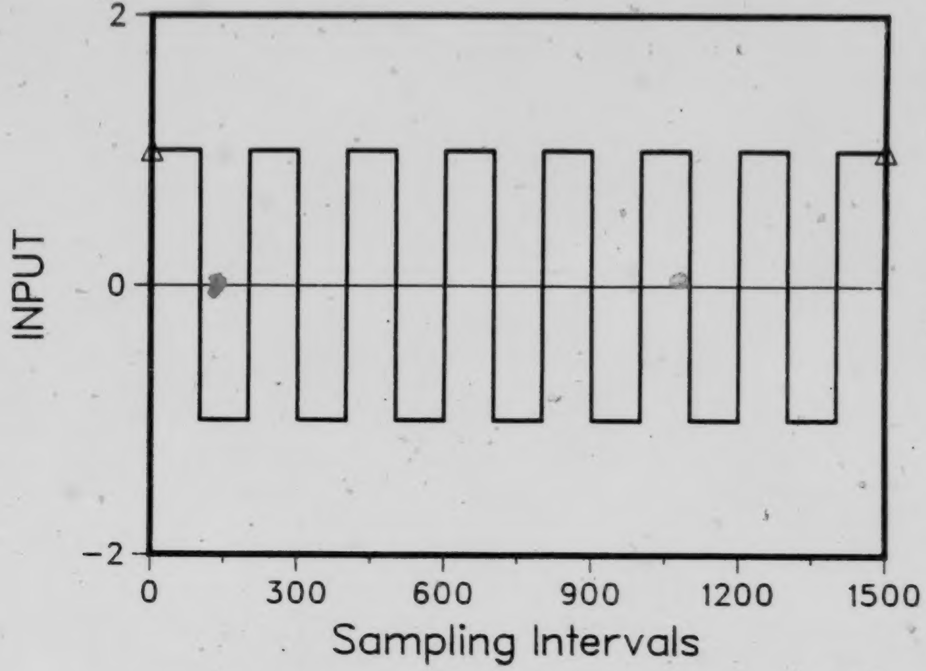


Fig.—2.17 Process input trajectory

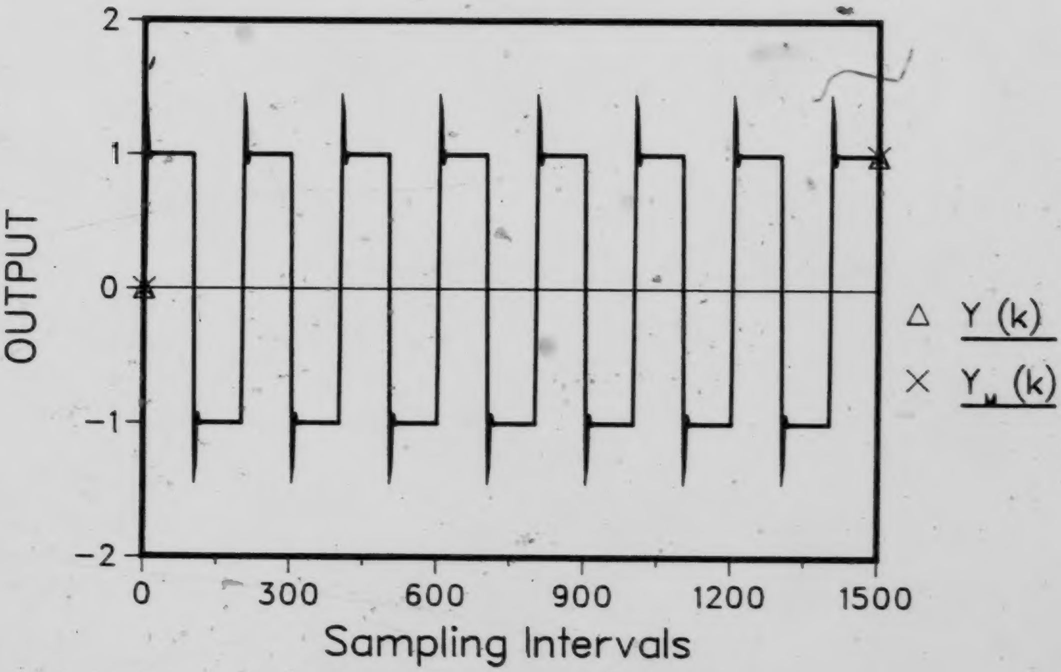


Fig.—2.18 Process and Model output trajectories
N_0=10 P(0)=1.E6 λ_{max}=0.9

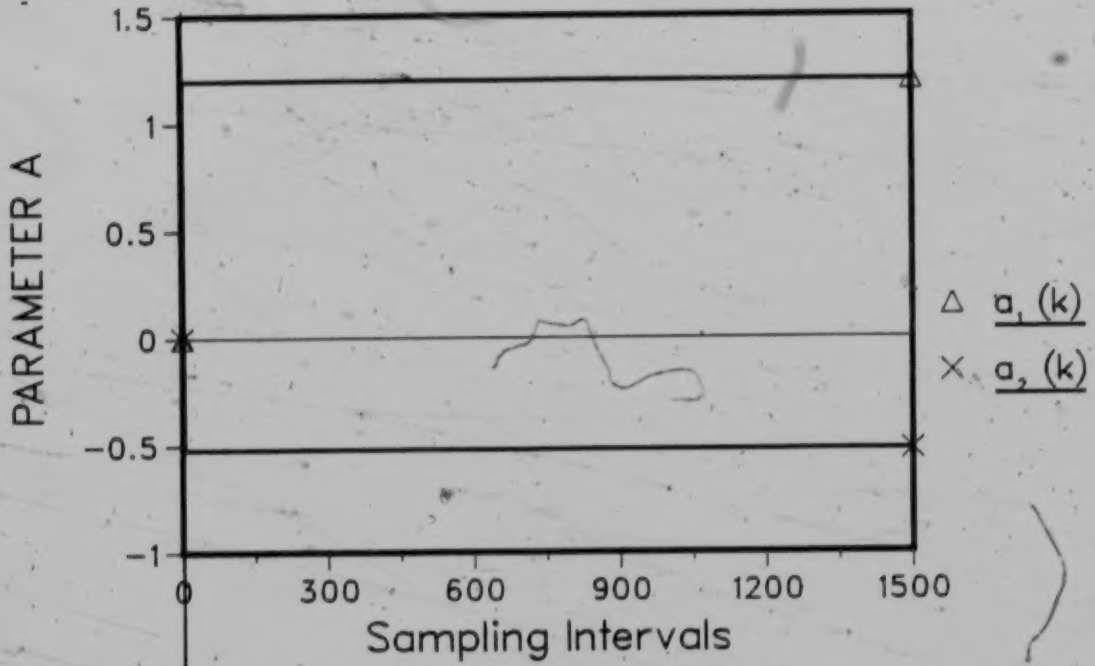


Fig.−2.19 Process parameter trajectory

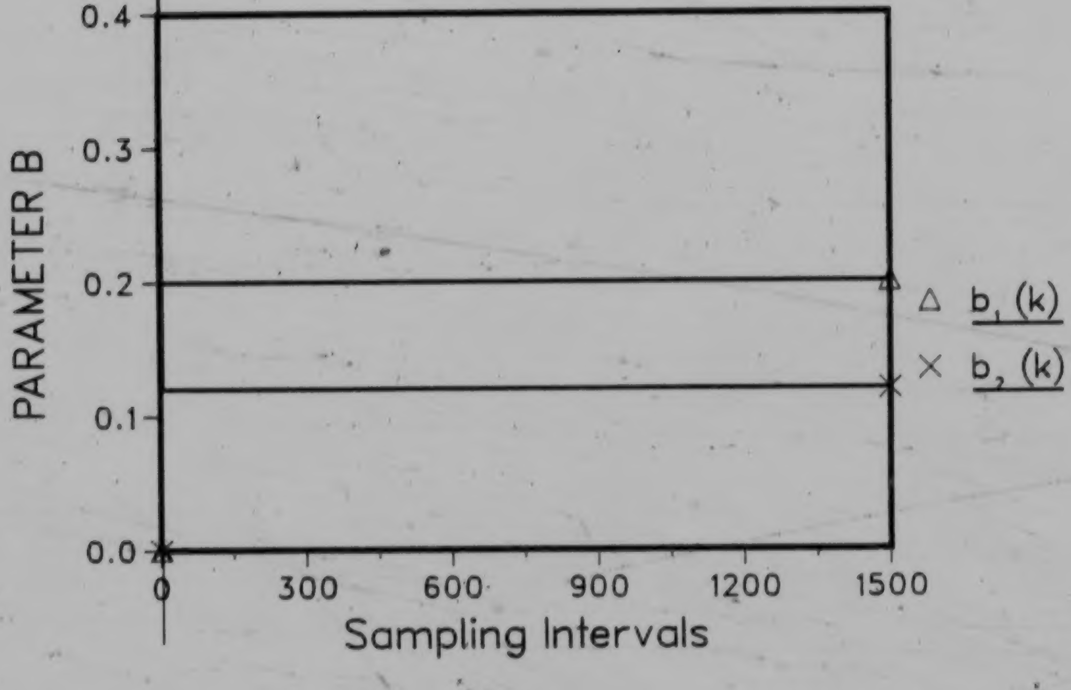


Fig.−2.20 Process parameter trajectory
$N_0=10$ $P(0)=1.E6$ $\lambda_{max}=0.9$

2.6 Conclusions

In this chapter we have presented the identification scheme which will be used in the rest of this work. Initially we presented its derivation and justified the need in using a variable forgetting factor. Subsequently we established its usefulness by proving properties P1 to P2. A similar work, for a closed loop system, has been done by Cordero and Mayne (1981) but they require the covariance matrix to be bounded in order to prove the stability of the identification scheme. In our work, we have removed this assumption. After justifying the need for exponential convergence of an identification scheme, we proved that the identification scheme used in this work is exponentially convergent if the regressor is persistently exciting and the forgetting factor is less than one. The analysis on exponential convergence led us to the conclusion that the "blow up" problem can be avoided if persistently exciting signals are used. Despite its complexity, due to the variable forgetting factor, this analysis can be considered as a direct extension of Johnstone's results in the case that a variable forgetting factor is used. Also we tried to give an insight into the definition of a persistently exciting signal. Finally, simulation results illustrating the performance of the identification scheme have been presented.

3. Pole-assignment control algorithm (PACA)

3.1 Introduction

The conventional three term (PID) controller is unquestionably the most common regulator in industry. Despite this, there are many cases where this simple controller cannot perform very well. There are processes where it is more advantageous to use more complex controllers. The time varying nature of many processes is the main reason for using more complex controllers such as adaptive controllers. Since its introduction (Kalman (1958)) the area of adaptive control has grown substantially. A number of different adaptive control strategies has been introduced. All these controllers can be separated into two classes from a design point of view. One class is characterized by the direct method of design, while the other one by the indirect method. The basic configuration of a direct method can be seen in figure 3.1.

According to this design procedure the controller parameters are directly estimated and used to calculate the gains of the control block. The estimation or identification block does not require any explicit knowledge of the process parameters. This method which is also called implicit design method (mostly by the European authors) was originally proposed by Whitaker in 1958. The MRAS and APCS controllers are two representatives of this design method.

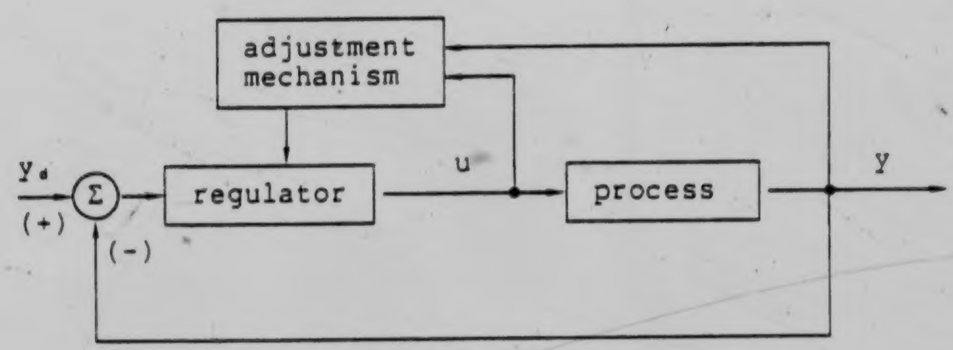

adjustment mechanism
regulator
process
u
y
y_d
Σ
(+)
(-)

Figure 3.1 - General Structure of Direct Method

The basic structure of the indirect or explicit method is as follows:

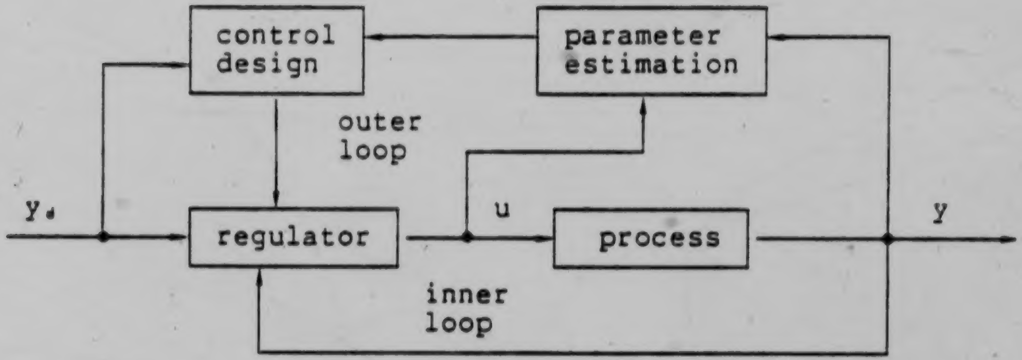

control design
parameter estimation
outer loop
regulator
process
inner loop
u
y
y_d

Figure 3.2 - General Structure of Indirect Method

According to this design method, the process parameters are estimated explicitly. From estimated process parameters the controller gains can be calculated (indirect process of calculation). The self-tuning regulators (Aström and Wittenmark (1983)) are an example of this design method.

It is usually very difficult to argue which is the best design method. Obviously, in the indirect method since we know the process parameters it is easy to construct the control law. On the other hand, the direct method is computationally more efficient. Despite the design method, the stability analysis of the closed loop system, under adaptive control, was and still remains a real problem. Stability has been proven for a class of controllers but under some assumptions. One usual assumption is that the process is assumed to be stably invertible i.e. be minimum phase. Since many practical processes exhibit some form of nonminimum phase behavior depending on the choice of the sampling time, this appeared to restrict the usefulness of adaptive control. However, during the last few years new algorithms have been suggested (Edmunds (1976); Wellstead et.al. (1979a, 1979b); Allindina and Hughes (1980); Aström (1980), Aström and Wittenmark (1980), Goodwin and Sin (1981), Elliott (1982), Elliott et.al.(1983), Clarke (1982), Anderson and Johnstone (1983) to name a few) which partly overcome these difficulties and have been shown to be effective in many cases.

The pole-assignment design method was quite well known a long time ago (Ragazzini and Franklin (1958)). But the first suggested use of explicit pole-assignment controllers was in a thesis by Edmunds (1976). This work was further developed by Wellstead et.al. (1979a, 1979b) for the stochastic case. Deterministic servo designs were considered in Aström(1980) and Aström and Wittenmark (1980). Goodwin and Sin (1981) suggested a quite simple algorithm for the deterministic case. Local stability results have also been presented in their work. Elliott (1982) presented a direct adaptive pole-assignment algorithm for nonminimum phase systems without any stability proof. Elliott et.al. (1982) presented similar results but for the multivariable case. In 1983, Elliott et.al. (1983) presented a hybrid adaptive controller for assigning the closed loop poles of a single input/single output continuous time system. The resulting closed loop system was shown to be globally stable when the reference signal was persistently exciting. It must be mentioned that their stability analysis is valid only for sinusoidal excitation and this is a significant drawback. The second drawback is related to the sampling and parameter update scheme. A key to the proof was the use of a special hybrid sampling and control parameter update scheme. It required that the control parameters remain constant while $N \geq 9n-1$ samples were processed (n is the order of the open-loop system). Obviously this constraint limits the speed of adaptation. The pole-assignment algorithm presented by Anderson and Johnstone (1983) is a version of the algorithm of Goodwin and Sin (1981). The whole design is based on the deterministic case and the structure of the control law is such that global stability can be proven if a number of fixed controllers and a persistently exciting signal are used.

In this work, the pole-assignment algorithm presented by Goodwin and Sin (1983) will be considered since it is **very simple** and uses the classical approach (it can be found in many texts) where the control objective is to locate the closed loop poles at prespecified locations. Our **objective** is to extend the stability analysis presented by Goodwin and Sin (1981) to the stochastic case. The theory relative to this control algorithm is presented in the next section.

3.2 Pole-assignment Algorithm

Consider a single input/single output linear discrete time-invariant system which is corrupted by a bounded disturbance and is described by

$$A(z^{-1})\ y(k) = B(z^{-1})\ u(k) + n(k) \tag{3.1}$$

where z^{-1} is the time delay operator, $\{u(k)\}$ and $\{y(k)\}$ are the input and output sequences, $n(k)$ is the bounded disturbance at the discrete time instant k. The polynomials $A(z^{-1})$ and $B(z^{-1})$ are given by

$$A(z^{-1}) = 1 - a_1\ z^{-1} - \dots - a_n\ z^{-n} \tag{3.2}$$

$$B(z^{-1}) = b_{d+1}\ z^{-d-1} + \dots + b_{d+m}\ z^{-(m+d)} \tag{3.3}$$

where d is the time delay ($d \geq 0$). Suppose that we try to control the system by using the following control law

$$L(z^{-1})\ u(k) = P(z^{-1})\ [y^*(k) - y(k)] \tag{3.4}$$

where $L(z^{-1})$ and $P(z^{-1})$ are polynomials in the z^{-1} operator representing the denominator and numerator, respectively, of the controller transfer function while $y^*(k)$ is the reference input signal.

The closed loop control configuration is as follows:

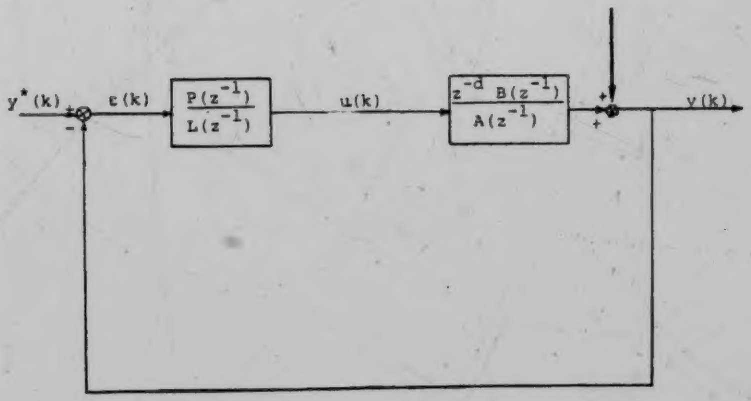

$y^*(k)$
+
-
$\varepsilon(k)$
$\frac{P(z^{-1})}{L(z^{-1})}$
$u(k)$
$\frac{z^{-d}B(z^{-1})}{A(z^{-1})}$
+
+
$y(k)$

Figure 3.3 - Closed Loop Configuration

In figure 3.3, $\varepsilon(k)$ is the tracking error defined by

$$\varepsilon(k) = y^*(k) - y(k) \tag{3.5}$$

while

$$B_1(z^{-1}) = b_{d+1} z^{-1} + \ldots + b_{d+m} z^{-(m+d)} \quad (3.6)$$

We can easily find that the transfer function between y(k) and $y^*(k)$ is given by

$$\frac{y(k)}{y^*(k)} = \frac{P(z^{-1})\, z^{-d}\, B_1(z^{-1})}{L(z^{-1})\, A(z^{-1}) + P(z^{-1})\, z^{-d}\, B_1(z^{-1})} \quad (3.7)$$

At least for the deterministic case, if our objective were to have the system output, y(k), follow the reference signal $y^*(k)$ exactly this would mean that

$$L(z^{-1})\, A(z^{-1}) + P(z^{-1})\, z^{-d}\, B_1(z^{-1}) = P(z^{-1})\, z^{-d}\, B_1(z^{-1}) \quad (3.8)$$

Equality (3.8) means that some poles of the closed loop system have been assigned to the open loop zeros. In this case, the closed loop system will have unstable poles if the zeros of $B_1(z^{-1})$ are inside the unit circle i.e. if the system is nonminimum phase. The idea of pole-assignment control is to move the closed loop poles to desired locations such that the closed loop system is stable. If the location of the desired closed loop poles is described by a stable polynomial $A^*(z^{-1})$, then we have to estimate the controller by solving the following equation:

$$L(z^{-1})\, A(z^{-1}) + P(z^{-1})\, z^{-d}\, B_1(z^{-1}) = A^*(z^{-1}) \quad (3.9)$$

The question now is if we can always solve equation (3.9).

The following lemma, taken from Wolovich (1974), guarantees solution of equation (3.9) under some assumptions.

Lemma 3.1

If $A(z^{-1})$ and $B(z^{-1})$ are relatively prime polynomials of the form

$$A(z^{-1}) = 1 - a_1 z^{-1} - \dots - a_n z^{-n} \quad ; \quad a_n \neq 0$$

$$B(z^{-1}) = b_{d+1} z^{-1-d} + \dots + b_{d+m} z^{-(m+d)} \quad ; \quad b_{d+m} \neq 0$$

then there exist unique polynomials $L(z^{-1})$ and $P(z^{-1})$ of the form

$$L(z^{-1}) = l_0 + l_1 z^{-1} + \dots + l_{r-1} z^{-r+1}$$

$$P(z^{-1}) = l_0 + l_1 z^{-1} + \dots + l_{r-1} z^{-r+1}$$

with $r = \max\{n, m + d\}$ such that

$$L(z^{-1}) A(z^{-1}) + P(z^{-1}) B(z^{-1}) = A^*(z^{-1})$$

where $A^*(z^{-1})$ is an arbitrary monic polynomial of degree $2r-1$.

Proof

See Wolovich (1974).

In the general case where $r = \max\{n, m+d\}$, equation (3.9) can be written as

$$\begin{bmatrix} 1 & & & 0 & & \\ a_1 & 1 & & b_1 & 0 & \\ \vdots & a_1 & & \vdots & b_1 & \\ \vdots & \vdots & 1 & \vdots & \vdots & 0 \\ a_{r-1} & \vdots & a_1 & b_{r-1} & \vdots & b_1 \\ & a_{r-1} & \vdots & & b_{r-1} & \vdots \\ & & a_{r-1} & & & b_{r-1} \end{bmatrix} \begin{bmatrix} \ell_0 \\ \vdots \\ \ell_{r-1} \\ p_0 \\ \vdots \\ p_{r-1} \end{bmatrix} = \begin{bmatrix} 1 \\ a_1^* \\ \vdots \\ \vdots \\ a_{2r-1}^* \end{bmatrix} \qquad (3.10)$$

Pole-assignment equation in matrix-vector form.

Equation (3.10) can be solved analytically by using different techniques such as Gaussian elimination or Decomposition method. The uniqueness of its solution is guaranteed by the above lemma (Sylvester's theorem) since $A(z^{-1})$ and $B(z^{-1})$ are prime polynomials.

In ~~a~~ closed loop system since we do not know the true process parameters, we have to estimate them on line. Then, accordng to the Certainty equivalence principle, we use them, as if they were the true process parameters, to calculate the controller parameters by solving equation (3.9). The identification scheme analysed in the previous chapter is used for the identification process. Equation (3.10) is solved analyticaly by using the decomposition method.

In summary we have

Process:

$$y(k) = \underline{\phi}(k-1)^T \underline{\theta_0} + n(k) \qquad (3.11)$$

Model:

$$\hat{y}(k) = \underline{\phi}(k-1)^T \, \underline{\hat{\theta}}(k-1) \qquad (3.12)$$

Prediction error:

$$e(k) = \underline{\phi}(k) - \hat{y}(k) \qquad (3.13)$$

Gain:

$$K(k) = \frac{\underline{P}(k-1) \, \underline{\phi}(k-1)}{1 + \underline{\phi}(k-1)^T \, \underline{P}(k-1) \, \underline{\phi}(k-1)} \qquad (3.14)$$

Parameters:

$$\underline{\hat{\theta}}(k) = \underline{\hat{\theta}}(k-1) + \underline{K}(k) \, e(k) \qquad (3.15)$$

Forgetting Factor:

$$\lambda(k) = \lambda_{max} - \frac{e(k)^2}{[\sigma_0^2 + \underline{\phi}(k-1)^T \underline{P}(k-1) \underline{\phi}(k-1)] \Sigma_0} \qquad (3.16)$$

or

$$\lambda(k) = \lambda_{min} \text{ if } \lambda(k) < \lambda_{min}$$

$$\Sigma_0 = \sigma_0^2 \, N_0$$

Covariance Matrix:

$$\underline{P}(k-1) = \frac{1}{\lambda(k)} \left[\underline{P}(k-1) - \frac{\underline{P}(k-1) \, \underline{\phi}(k-1)^T \, \underline{P}(k-1)}{1 + \underline{\phi}(k-1)^T \, \underline{P}(k-1) \, \underline{\phi}(k-1)}\right] \qquad (3.17)$$

Pole-assignment equation:

$$\hat{L}(z^{-1},k) \, u(k) = \hat{P}(z^{-1},k) \, [y^*(k) - y(k)] \qquad (3.18)$$

Control law:

$$\hat{L}(z^{-1},k) \, u(k) = \hat{P}(z^{-1},k) \, [y^*(k) - y(k)] \qquad (3.19)$$

where

$$\underline{\phi}(k-1)^T = [y(k-1) \;\ldots\; y(k-r) \;\; u(k-1) \;\ldots\; u(k-r)] \quad (3.20)$$

$$\underline{\theta}_0^T = [a_1 \;\ldots\; a_r \;\; b_1 \;\ldots\; b_r] \quad (3.21)$$

with

$a_i = 0$ if $i > n$

$b_i = 0$ if $i < d+1$ (this requires knowledge of d)

$b_i = 0$ if $i > m+d$

$$\underline{\hat{\theta}}(k)^T = [\hat{a}_1(k) \;\ldots\; \hat{a}_r(k) \;\; \hat{b}_1(k) \;\ldots\; \hat{b}_r(k)] \quad (3.22)$$

$$\hat{A}(z^{-1},k) = 1 - \hat{a}_1(k)\, z^{-1} - \ldots - \hat{a}_r(k)\, z^{-r} \quad (3.23)$$

$$\hat{B}(z^{-1},k) = \hat{b}_1(k)\, z^{-1} - \ldots - \hat{b}_r(k)\, z^{-r} \quad (3.24)$$

$$\hat{L}(z^{-1},k) = \hat{l}_0(k) + \hat{l}_1(k)\, z^{-1} + \ldots + \hat{l}_{r-1}(k)\, z^{-(r-1)} \quad (3.25)$$

$$\hat{P}(z^{-1},k) = \hat{P}_0(k) + \hat{P}_1(k)\, z^{-1} + \ldots + \hat{P}_{r-1}(k)\, z^{-(r-1)} \quad (3.26)$$

It can be easily seen that the on-line implementation of this control algorithm presents some problems. Even if the polynomials $A(z^{-1})$ and $B(z^{-1})$ are prime, there is no guarantee that the polynomials $\hat{A}(z^{-1}, k)$ and $\hat{B}(z^{-1}, k)$ will be prime for all k. For the time instants when these are not prime, equation (3.18) will not have a unique solution. One way of avoiding this problem is to assume (Goodwin and Sin

(1981)) that our initial parameter vector $\underline{\theta}(0)$ is within a region, in the parameter vector space, such that equation (3.18) is always solvable. Of course, this requires good apriori knowledge of the process under control.

Based on this approach they proved local stability of the closed loop system for the deterministic case. In the next section, we shall use the same assumption to prove local stability (boundness of {u(k)} and {(k)}) in the presence of bounded disturbances. We shall also discuss the possibility of removing this assumption and then conjecture/speculate on global stability results.

3.3 Stability Analysis

In this section, we shall prove local stability of the control algorithm proposed in section 3.2. We shall use different ways to approach this problem in the deterministic and the stochastic cases. The reason for this is that the stability analysis is based on the convergence of the estimates of the process parameters. The identification scheme as it was proposed in the second chapter can only guarantee convergence of the estimates of the process parameters in the noise-free case. If we still want convergence of the parameter estimates in the stochastic case, we have to modify the identification scheme. The idea is to stop adaptation when the prediction error e(k) becomes less than a function of the upper bound of the noise. This idea has been introduced by Martin-Sanchez (1983) who used a projection algorithm. For the deterministic case, the identification scheme given by equations (3.14) to (3.17) will be used. In the stochastic case, equations (3.14) and (3.17) will be substituted by the following equations.

$$\underline{K}(k) = \frac{\rho(k)\ \underline{P}(k-1)\ \underline{\psi}(k-1)}{1 + \rho(k)\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \tag{3.27}$$

and

$$\underline{P}(k) = \frac{1}{\lambda(k)}\left[\underline{P}(k-1) - \frac{\rho(k)\ \underline{P}(k-1)\ \underline{\psi}(k-1)\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)}{1 + \rho(k)\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)\ \underline{\psi}(k-1)}\right] \tag{3.28}$$

respectively. The scalars $\rho(k)$, which are positive numbers, are defined below and used to determine a criterion for stopping or continuing parameter adaptation which is essential for the proof of stability. We define.

a] $\rho(k) = 0$ if and only if

$$|e(k)| \leq \Delta(\rho_1, \Delta_b, k) \leq 2\Delta_b < \infty \tag{3.29}$$

and

$$\Delta(\rho_1, \Delta_b, k) = \frac{2 + 2\rho_1\ \underline{\psi}(k-1)^T \underline{P}(k-1)\ \underline{\psi}(k-1)}{2 + \rho_1\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)\ \underline{\psi}(k-1)}\ \Delta_b \tag{3.30}$$

$$\text{with}\quad 0 < \rho_1 < \infty \quad\text{and}\quad \Delta_b \geq \max_{0 \leq k \leq \infty} |n(k)| \tag{3.31}$$

where ρ_l is a lower bound for the scalars $\rho(k)$ and Δb is an estimate of a constant upper bound on the absolute value of the bounded noise, n(k), affecting the system.

b] $\rho_l < \rho(k) < \rho_b(k) < \rho_u < \infty$ if and only if

$$|e(k)| > \Delta(\rho_l, \Delta_b, k) > \Delta_b \tag{3.32}$$

where $\rho b(k)$ is defined as follows:

1] $\rho_b(k) = \rho_u$ if $|e(k)| > \Delta(\rho_u, \Delta_b, k)$ (3.33)

2] $$\rho_b(k) = \frac{2[|e(k)| - \Delta_b]}{[2\Delta_b - |e(k)|]\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1)} \tag{3.34}$$

if $\Delta(\rho_l, \Delta_b, k) < |e(k)| < \Delta(\rho_u, \Delta_b, k)$ (3.35)

According to the definition of $\rho b(k)$, in equations (3.33) and (3.34), it can be proven (Martin-Sanchez (1983)) that for all $\rho(k) \neq 0$, the following condition is verified:

$$|e(k)| > \Delta(\rho(k), \Delta_b, k) = \frac{2 + 2\rho(k)\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1)}{2 + \rho(k)\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1)}\ \Delta_b \tag{3.36}$$

The above modified least squares identification algorithm will stop adaptation when $|e(k)|$ is less than or equal to $\Delta(\rho_l, \Delta_b, k)$. During periods when $|e(k)|$ is greater than $\Delta(\rho_l, \Delta_b, k)$, the scalar $\rho(k)$ is chosen in such a way that inequality (3.36) is satisfied. For this particular identification scheme, we can prove that the norm of the parameter error vector will be decreasing function over time in the sense that

$$\underline{\tilde{\theta}}(k)^T \underline{P}(k-1)^{-1} \underline{\tilde{\theta}}(k) < \underline{\tilde{\theta}}(k-1)^T \underline{P}(k-1)^{-1} \underline{\tilde{\theta}}(k-1)$$

for all $k \geq 1$, if the adaptation is on.

Lemma 3.2

Along the solution of the modified least squares algorithm

$$\underline{\tilde{\theta}}(k)^T \underline{P}(k-1)^{-1} \underline{\tilde{\theta}}(k) < \underline{\tilde{\theta}}(k-1)^T \underline{P}(k-1)^{-1} \underline{\tilde{\theta}}(k-1) \qquad \forall\ k > 1 \tag{3.37}$$

during periods of adaptation.

Proof:

We define the a posteriori prediction error $e(k/k)$ as

$$e(k/k) = y(k) - \underline{\phi}(k-1)^T \underline{\hat{\theta}}(k) = \underline{\phi}(k-1)^T \underline{\tilde{\theta}}(k) + n(k) \tag{3.38}$$

Equations (3.12), (3.13) and (3.38) give that

$$e(k/k) - e(k) = \underline{\phi}(k-1)^T [\underline{\hat{\theta}}(k-1) - \underline{\hat{\theta}}(k)]$$

or

$$e(k/k) = \frac{e(k)}{1 + \rho(k)\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1)} \tag{3.39}$$

on using equation (3.15).

From equations (3.15), (3.27) and (3.39), we get that

$$\underline{\hat{\theta}}(k) = \underline{\hat{\theta}}(k-1) + \rho(k)\ \underline{P}(k-1)\ \underline{\phi}(k-1)\ e(k/k) \tag{3.40}$$

which means that

$$\underline{\tilde{\theta}}(k) + \rho(k)\ \underline{P}(k-1)\ \underline{\phi}(k-1)\ e(k/k) = \underline{\tilde{\theta}}(k-1) \qquad (3.41)$$

By multiplying eqaution (3.41) by $\underline{\theta}(k)\ \underline{P}(k-1)^{-1}$, we get

$$\underline{\tilde{\theta}}(k)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k) + \rho(k)\ \underline{\tilde{\theta}}(k)^T\ \underline{\phi}(k-1)\ e(k/k) =$$
$$\underline{\tilde{\theta}}(k)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k-1)$$

or

$$\underline{\tilde{\theta}}(k)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k) = [\underline{\tilde{\theta}}(k-1)^T - \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \rho(k)\ e(k/k)]$$
$$\underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k-1) - \rho(k)\ e(k/k)\ \underline{\tilde{\theta}}(k)^T\ \underline{\phi}(k-1)$$
$$= \underline{\tilde{\theta}}(k-1)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k-1) - \rho(k)\ e(k/k)\ \underline{\phi}(k-1)^T\ \underline{\tilde{\theta}}(k-1) -$$
$$\rho(k)\ e(k/k)\ \underline{\phi}(k-1)^T\ \underline{\tilde{\theta}}(k)$$
$$= \underline{\tilde{\theta}}(k-1)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k-1) - 2\rho(k)\ e(k/k)\ \underline{\phi}(k-1)^T\ \underline{\tilde{\theta}}(k) -$$
$$\rho(k)^2\ e(k/k)^2\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1) \qquad (3.42)$$

Since we require:

$$\underline{\tilde{\theta}}(k)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k) < \underline{\tilde{\theta}}(k-1)^T\ \underline{P}(k-1)^{-1}\ \underline{\tilde{\theta}}(k-1)$$

equation (3.42) implies that

$$-2e(k/k)\ \underline{\phi}(k-1)^T\ \underline{\tilde{\theta}}(k) - \rho(k)\ e(k/k)^2\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1) < 0$$

or

$$-2[e(k/k) - n(k)]\ e(k/k) - \rho(k)\ e(k/k)^2\ \underline{\phi}(k-1)^T$$
$$\underline{P}(k-1)\ \underline{\phi}(k-1) < 0$$

or $2|n(k)| < |e(k/k)|\ [2 + \rho(k)\ \underline{\phi}(k-1)^T\ \underline{P}(k-1)\ \underline{\phi}(k-1)]$

or

$$|e(k/k)| > \frac{2|n(k)|}{2 + \rho(k)\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)\ \underline{\psi}(k-1)} \qquad (3.43)$$

From (3.39) and (3.43) we get

$$|e(k)| > \frac{2 + 2\rho(k)\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)\ \underline{\psi}(k-1)}{2 + \rho(k)\ \underline{\psi}(k-1)^T\ \underline{P}(k-1)\ \underline{\psi}(k-1)}\ |n(k)| \qquad (3.44)$$

Inequality (3.44) is valid since condition (3.36) is always valid for every $\rho(k) \neq 0$. This proves lemma 3.2.

If we assume that we also know the minimum value, Δ_m, of the upper bound Δb and that

$$\Delta_b - \Delta_m = \delta \qquad \text{where } \delta > 0$$

$$\Delta_m = \max_{1 \leq k \leq \infty} |n(k)|$$

we can follow the same lines as in Martin-Sanchez (1983) to prove that if the number of time instants for which $\rho(k) \neq 0$, tends to infinity, then

$$\lim_{k \to \infty} e(k/k) = 0$$

and

$$\lim_{k \to \infty} ||\hat{\underline{\theta}}(k) - \hat{\underline{\theta}}(k-h)|| = 0 \qquad \text{for any finite } h$$

The latter implies asymptotic convergence of the parameter estimates in the limit.

Having proven these preliminary results, we shall proceed in proving local stability of the closed loop system under pole-assignment control. The problem related to the solvability of the pole-assignment equation (3.18), during the transient state of the system, can be avoided if we assume good knowledge about the process parameters. The following lemma, presented in Goodwin and Sin (1981), overcomes the solvability problem of equation (3.18) during the transient state.

Lemma 3.3

There exists a region in the parameter space centered on $\underline{\theta}_0$ and having radious $g \geq 0$, such that if $\underline{\theta}_0$ is within this region

$||\underline{\theta}(0) - \underline{\theta}_0|| \leq g$

then equation (3.18) is solvable for all k and both $\hat{L}(z^{-1},k)$ and $\hat{P}(z^{-1},k)$ have bounded coefficients.

Proof:

See proof in Goodwin and Sin (1981).

Our stability result will be based on the following assumptions.

A1] $r = \max\{n, m+d\}$ is known

A2] $n(k)$ is bounded noise for all k with Δb an upper bound.

A3] $y^*(k)$ is a bounded reference signal for all k.

A4] $A^*(z^{-1})$ is a stable monic polynomial of degree $2r-1$.

A5] Lemma 3.2 is satisfied.

A6] Process parameter adaptation is turned on or off according to the modified least squares identification algorithm.

A7] A fixed gain controller able to stabilize the system is used when the parameter adaptation is on.

Based on these assumptions, the following theorem can be proven.

Theorem 3.1

If the assumptions A1 to A7 are satisfied the algorithm (3.11) to (3.26) is stable in the sense that

a] $\{u(k)\}$ is bounded

b] $\{y(k)\}$ is bounded

c] The fixed gain controller will be used only finitely often.

Proof:

We define

$$\hat{A}\hat{B} = \sum_i \sum_j \hat{a}_i(k)\, \hat{b}_j(k)\, z^{-i-j} = \hat{B}\hat{A} \tag{3.45}$$

$$\hat{A}\cdot\hat{B} = \sum_i \sum_j \hat{a}_i(k)\, \hat{b}_j(k-1)\, z^{-i-j} \neq \hat{B}\cdot\hat{A} \tag{3.46}$$

$$\hat{\bar{B}} = \hat{B}\,(z^{-1}, k-1) \tag{3.47}$$

Note that $\hat{A}\hat{B} = \hat{A}\cdot\hat{B} = \hat{B}\hat{A}$

when $\hat{A}$, $\hat{B}$ are time invariant

We also have

$$Ay(k) = Bu(k) + n(k) \tag{3.48}$$

$$\hat{L}u(k) = \hat{P}y^{*}(k) - \hat{P}y(k) \tag{3.49}$$

$$\hat{A}\hat{L} + \hat{B}\hat{P} = A^{*} \tag{3.50}$$

The prediction error, e(k), is defined in equation (3.13). We have

$$\begin{aligned} e(k) &= y(k) - \hat{y}(k) \\ &= y(k) - \underline{\phi}(k-1)^{T}\, \underline{\hat{\theta}}(k-1) \\ &= y(k) - \hat{a}_1(k-1)\, y(k-1) - \dots - \hat{a}_r(k-1)\, y(k-r) - \\ &\qquad \hat{b}_1(k-1)\, u(k-1) - \dots - \hat{b}_r(k-1)\, u(k-r) \\ &= [1-\hat{a}_1(k-1)\, z^{-1} - \dots - \hat{a}_r(k-1)z^{-r}]\; y(k) - \\ &\qquad [\hat{b}_1(k-1)\, z^{-1} + \dots + \hat{b}_r(k-1)z^{-r}]\; u(k) \end{aligned}$$

or

$$e(k) = \hat{\bar{A}}y(k) - \hat{\bar{B}}u(k) \tag{3.51}$$

We also define the auxiliary signal w(k) as

$$w(k) = \hat{A} \bullet \hat{P} y^{*}(k) \tag{3.52}$$

Equation (3.52) gives that

$$\begin{aligned}
w(k) &= \hat{A} \bullet \hat{P} y^{*}(k) \\
&= \hat{A} \bullet \hat{L} u(k) + \hat{A} \bullet \hat{P} y(k) \\
&= \hat{A}\ \hat{L} u(k) + [\hat{A} \bullet \hat{L} - \hat{A}\hat{L}] u(k) + \hat{A}\ \hat{P} y(k) + \\
&\qquad [\hat{A} \bullet \hat{P} - \hat{A}\hat{P}] y(k) \\
&= \hat{A}\ \hat{L} u(k) + \hat{B}\ \hat{P} u(k) + \hat{P} e(k) + [\hat{A} \bullet \hat{L} - \hat{A}\hat{L}] u(k) + \\
&\qquad [\hat{A} \bullet \hat{P} - \hat{A}\hat{P}] y(k) - \hat{B}\hat{P} u(k) - \hat{P} e(k) + \hat{A}\hat{P} y(k) \\
&= \hat{A}\hat{L} u(k) + \hat{B}\hat{P} u(k) + \hat{P} e(k) + [\hat{A} \bullet \hat{L} - \hat{A}\hat{L}] u(k) + \\
&\qquad [\hat{A} \bullet \hat{P} - \hat{A}\hat{P}] y(k) - \hat{P}\hat{B} u(k) - \hat{P} \bullet [\hat{\bar{A}} y(k) - \\
&\qquad \hat{\bar{B}} u(k)] + \hat{P}\hat{A} y(k) \\
&= \hat{A}\hat{L} u(k) + \hat{B}\hat{P} u(k) + \hat{P} e(k) + [\hat{A} \bullet \hat{L} - \hat{A}\hat{L}] u(k) + \\
&\qquad [\hat{A} \bullet \hat{P} - \hat{A}\hat{P}] y(k) + [\hat{P} \bullet \hat{\bar{B}} - \hat{P}\hat{B}] u(k) - \\
&\qquad [\hat{P} \bullet \hat{\bar{A}} - \hat{P}\hat{A}] y(k)
\end{aligned}$$

or

$$\begin{aligned}
w(k) &= A^{*}u(k) + \hat{P}e(k) + [\hat{A}\cdot\hat{L} - \hat{A}\hat{L}]u(k) + [\hat{A}\cdot\hat{P} - \hat{A}\hat{P}]\ y(k) + \\
&[\hat{P}\cdot\hat{\bar{B}} - \hat{P}\hat{B}]u(k) - [\hat{P}\cdot\hat{\bar{A}} - \hat{P}\hat{A}]y(k) \qquad (3.53)
\end{aligned}$$

We define the auxiliary signal x(k) as

$$x(k) = \hat{B}\cdot\hat{P}y^{*}(k) \qquad (3.54)$$

Equation (3.54) gives

$$\begin{aligned}
x(k) &= \hat{B}\cdot\hat{P}y^{*}(k) \\
&= \hat{B}\cdot\hat{L}u(k) + \hat{B}\cdot\hat{P}y(k) \\
&= \hat{B}\hat{L}u(k) + [\hat{B}\cdot\hat{L} - \hat{B}\hat{L}]u(k) + \hat{B}\hat{P}y(k) + \\
&\qquad [\hat{B}\cdot\hat{P} - \hat{B}\hat{P}]y(k) \\
&= \hat{B}\hat{L}u(k) + \hat{A}\hat{L}y(k) - \hat{L}e(k) + [\hat{B}\cdot\hat{L}-\hat{B}\hat{L}]u(k) + \\
&\qquad \hat{B}\hat{P}y(k) + [\hat{B}\cdot\hat{P} - \hat{B}\hat{P}]y(k) - \hat{A}\hat{L}y(k) + \hat{L}e(k) \\
&= \hat{A}\hat{L}y(k) + \hat{B}\hat{P}y(k) - \hat{L}e(k) + [\hat{B}\cdot\hat{L} - \hat{B}\hat{L}]u(k) + \\
&\qquad [\hat{B}\cdot\hat{P} - \hat{B}\hat{P}]y(k) - \hat{A}\hat{L}y(k) + \\
&\qquad \hat{L}\cdot[\hat{\bar{A}}y(k) - \hat{\bar{B}}u(k)] + \hat{B}\hat{L}u(k) \\
&= \hat{A}\hat{L}y(k) + \hat{B}\hat{P}y(k) - \hat{L}e(k) + [\hat{B}\cdot\hat{L} - \hat{B}\hat{L}]u(k) + \\
&\qquad [\hat{B}\cdot\hat{P} - \hat{B}\hat{P}]y(k) + [\hat{L}\cdot\hat{\bar{A}} - \hat{A}\hat{L}]y(k) - \\
&\qquad [\hat{L}\cdot\hat{\bar{B}} - \hat{L}\hat{B}]u(k)
\end{aligned}$$

or

$$\begin{aligned}
x(k) = A^{*}y(k) - \hat{L}e(k) + [\hat{B}\cdot\hat{L} - \hat{B}\hat{L}]u(k) + \\
[\hat{B}\cdot\hat{P} - \hat{B}\hat{P}]y(k) + [\hat{L}\cdot\hat{\bar{A}} - \hat{L}\hat{A}]y(k) - [\hat{L}\cdot\hat{\bar{B}} - \hat{L}\hat{B}]\ u(k)
\end{aligned} \qquad (3.55)$$

Equations (3.53) and (3.55) give that

$$\begin{bmatrix} w(k) - \hat{P}e(k) \\ \\ x(k) + \hat{L}e(k) \end{bmatrix} = \begin{bmatrix} A^* + [\hat{A}\cdot\hat{L} - \hat{A}\hat{L}] + [\hat{P}\cdot\bar{\hat{B}} - \hat{P}\hat{B}] & [\hat{A}\cdot\hat{P} - \hat{A}\hat{P}] - [\hat{P}\cdot\bar{\hat{S}} - \hat{P}\hat{A}] \\ \\ [\hat{B}\cdot\hat{L} - \hat{B}\hat{L}] - \hat{L}\cdot\bar{\hat{B}} - \hat{L}\hat{B}] & A^* + [\hat{B}\cdot\hat{P} - \hat{B}\hat{P}] + [\hat{L}\cdot\bar{\hat{A}} - \hat{L}\hat{A}] \end{bmatrix} \begin{bmatrix} u(k) \\ \\ y(k) \end{bmatrix} \tag{3.56}$$

Equation (3.56) can be regarded as a linear time varying dynamical system having inputs $\{w(k)\}$, $\{x(k)\}$ and $\{e(k)\}$ and outputs $\{u(k)\}$ and $\{y(k)\}$. Inequality (3.29) means that the polynomials $\hat{A}(z^{-1},k)$ and $\hat{B}(z^{-1},k)$ will have bounded coefficients for all k. Also g has been chosen as in Lemma 3.2. Then, $\hat{L}(z^{-1},k)$ and $\hat{P}(z^{-1},k)$ will have bounded coefficients for all k. During periods when the adaptation is off, the square brackets in equation (3.55) will be zero. In this case, equation (3.55) reduces to the following one

$$\begin{bmatrix} w(k) - \hat{P}e(k) \\ \\ x(k) + \hat{L}e(k) \end{bmatrix} = \begin{bmatrix} A^* & 0 \\ \\ 0 & A^* \end{bmatrix} \begin{bmatrix} u(k) \\ \\ y(k) \end{bmatrix} \tag{3.57}$$

The sequences $\{w(k)\}$ and $\{x(k)\}$ will be bounded since the polynomials $\hat{A}(z^{-1},k)$ and $\hat{B}(z^{-1},k)$, $\hat{L}(z^{-1},k)$ and $\hat{P}(z^{-1},k)$ have bounded coefficients while the sequence $\{y^*(k)\}$, by assumption, is bounded. Then the sequences $\{u(k)\}$ and $y\{(k)\}$

will not grow faster than linearly with respect to $e(k)$. This means that the sequence $\{||\underline{\psi}(k-1)||\}$ will be linearly bounded by $|e(k)|$.

Then,

$$||\underline{\psi}(k-1)|| < C_1 + C_2 \cdot \max_{0<\tau<k} |e(\tau)| \qquad (3.58)$$

where

$$0 < C_1 < \infty \quad \text{and} \quad 0 < C_2 < \infty$$

During periods when the parameter adaptation is off, we have

$$|e(k)| < 2\Delta_b$$

For such periods of time, inequality (3.58) implies that the sequences $\{u(k)\}$ and $\{y(k)\}$ are bounded. At the limit, since convergence of the parameter estimates is guaranteed by the identification scheme, the sequences $\{u(k)\}$ and $\{y(k)\}$ will also be bounded. If the adaptation is on, the fixed controller guarantees stability of the system, by assumption. The problem is how someone can shorten the time period over which the fixed gain controller is used. This can only be done, if the rate of the parameter convergence is increased. This implies that a persistently exciting reference signal must be used. In the deterministic case, the unmodified identification scheme of the second chapter can be used. This identification scheme guarantees properties P1 to P3. For this case, inequality (3.58) will also be valid. From equation (2.60) and inequality (3.58) we can conclude that the sequences $\{u(k)\}$ and $\{y(k)\}$ are bounded as well as the prediction error goes to zero, on using the Key technical lemma of Goodwin, Ramadge and Caines (1980). Still, the stability is asymptotic because property P2 is valid in the limit. It seems reasonable to seek for conditions to shorten the time period over which property P2 does not hold. In the next section we shall show that a persistently exciting reference signal, despite the presence of bounded noise or not, can lead to a persistently exciting regressor. For the least squares identification scheme with a forgetting factor less than one, it has been proven, in Chapter 2, that exponentially fast estimation of the true process parameters is possible. In such a case, the closed loop system will become exponentially stable if no disturbances affect it while in the stochastic case the total time period over which the fixed gain controller is used will be kept as low as possible.

3.4 Global Stability

The previous stability analysis is local in nature since our initial parameter vector $\underline{\theta}(0)$ must be close enough to the true one. Instead of having a good initial parameter vector $\underline{\theta}(0)$, suppose that we start controlling the process with a fixed controller and run the identification scheme in a passive mode. In the deteministic case, we shall turn to the adaptive pole-assignment controller only when the parameter estimates get into the region g of the parameter vector space mentioned in lemma 3.3. In the stochastic case maybe the time required to get into this region g is greater than that of the deteministic case. The use of persistently exciting regressor will shorten this initial period for both the deteministic and the stochastic cases. In a closed loop system, we can only manipulate the reference signal. If we prove that the regressor is persistently exciting when the reference signal is so, then the identification scheme will force the parameter estimates to get into the region g exponentially fast. In this case, the whole system will be globally asympotically stable. This will be obtained through the following two preliminary lemmas.

Lemma 3.4

Suppose that the system is described by equation (3.11), with $\underline{\psi}(k-1)$ and $\underline{\theta}_0$ appropriately defined, and that the control law is given by equation (3.19). Suppose that there exists $\gamma_0, \ldots, \gamma_{2r-1}$ not all zero, such that

$$\begin{bmatrix} \gamma_0 \ \gamma_1 \ \cdots \ \gamma_{2r-1} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-2r+1) \end{bmatrix} < C \qquad (3.59)$$

for all k $\epsilon[j+r, j+s+1]$

Then, there exists $\delta_0, \ldots, \delta_{4r-2}$, not all zero, such that

$$\begin{bmatrix}\delta_0 & \delta_1 & \cdots & \delta_{4r-2}\end{bmatrix} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} < \underline{\Omega}\, C + \xi\, M\, ||\gamma||\, (\Delta\theta)_{max} + 2r \max_{i\epsilon[0,2r-1]} \{|\gamma_i|\} \cdot 2r \max_{i\epsilon[1,r-1]} \{1, |\hat{l}_i(j+3r-1)|\} \cdot \Delta_b \qquad (3.60)$$

for all $k\ \epsilon[j+3r-1,\ j+s+1]$

where Ω and ξ are constants dependent on

$$r,\quad \max_{i\epsilon[1,r-1]} \{1_1, |\hat{l}_i(j+3r-1)|\},\quad \max_{i\epsilon[0,r-1]} \{|\hat{P}_i(j+3r-1)|\}$$

$$\max_{i\epsilon[1,r]} \{|b_i|\}, \max_{i\epsilon[1,r]} \{1, |a_i|\}$$

M is a constant which depends on the maximum values

$$\max_{k\epsilon[j-r,j+s]}\{|u(k)|\},\quad \max_{k\epsilon[j-r,j+s]}\{|y(k)|\}$$

$(\Delta\theta)$max depends on the maximum values of

$$|\hat{l}_i(k) - \hat{l}_i(j+3r-1)|$$

and

$$|\hat{P}_i(k) - \hat{P}_i(j+3r-1)| \quad (i=0, 1, \ldots, r-1) \quad (k=j, j+1, \ldots, j=s+1)$$

Proof

Inequality (3.59) can be written as

$$|\gamma(z^{-1})\ y\ (k)| < C$$

for all k $\epsilon[j+r,\ j+s+1]$ with

$$\gamma(z^{-1}) = \gamma_0 + \gamma_1 z^{-1} + \dots + \gamma_{2r-1} z^{-2r+1} \qquad (3.60a)$$

This implies thet for all k $\epsilon[j+3r-1,\ j+s+1]$

$$|\{\hat{L}(z^{-1}, j+3r-1)\, A(z^{-1}) - \hat{P}(z^{-1}, j+3r-1)\, B(z^{-1})\}\cdot$$

$$\gamma(z^{-1})\, y(k)| < |\hat{L}(z^{-1}, j+3r-1)\, A(z^{-1})\, \gamma(z^{-1})\, y(k)| +$$

$$|\hat{P}(z^{-1}, j+3r-1)\, B(z^{-1})\, \gamma(z^{-1})\, y(k)| < r\cdot\max_{i\varepsilon[1,r-1]}$$

$$\{1, |\hat{l}_i(j+3r-1)|\}\cdot(r+1)\cdot\max_{i\varepsilon[1,r]} \{1, |a_i|\}\cdot C + r\cdot\max_{i\varepsilon[0,r-1]}$$

$$\{|\hat{p}_i(j+3r-1)|\}\cdot r\cdot\max_{i\varepsilon[1,r]} \{|b_i|\}\cdot C$$

Also

$$B(z^{-1})\hat{P}(z^{-1},k)y^*(k) = B(z^{-1})\ [\hat{L}(z^{-1},k)u(k) - \hat{P}(z^{-1},k)\ y(k)] =$$

$$B(z^{-1})[\hat{L}(z^{-1},j+3r-1)u(k) - \hat{P}(z^{-1},j+3r-1)y(k)] +$$

$$B(z^{-1})\ [\hat{L}(z^{-1},k) - \hat{L}(z^{-1},j+3r-1)]u(k) - B(z^{-1})[\hat{P}(z^{-1},k) -$$

$$\hat{P}(z^{-1},j+3r-1)]y(k) = [\hat{L}(z^{-1},j+3r-1)\ A(z^{-1}) -$$

D

$$\hat{P}(z^{-1},j+3r-1)\ B(z^{-1})]y(k) - \hat{L}(z^{-1},j+3r-1)n(k) +$$

$$B(z^{-1})\ \{[\hat{L}(z^{-1},k) - \hat{L}(z^{-1},j+3r-1)]u(k) - [\hat{P}(z^{-1},k) -$$

$$\hat{P}(z^{-1},j+3r-1)]y(k)\}$$

Also, for $k \in [j+3r-1,\ j+s]$, we have

$$|\gamma_0\ B(z^{-1})\ \hat{P}(z^{-1},k)\ y^*(k) + \gamma_1\ B(z^{-1})\ \hat{P}(z^{-1},k)\ Y^*(k-1)$$

$$+\ldots+ \gamma_{2r-1}B(z^{-1})\ \hat{P}(z^{-1},k)\ y^*(k-2r+1)|$$

$$= |\gamma(z^{-1})\ B(z^{-1})\ \hat{P}(z^{-1},k)\ y^*(k)| <$$

$$< |\gamma(z^{-1})\ \{\hat{L}(z^{-1},j+3r-1)\ A(z^{-1}) - \hat{P}(z^{-1},j+3r-1)\ B(z^{-1})\}$$

$$y(k)| +$$

$$+ |\gamma(z^{-1})\ B(z^{-1})\ \{\hat{L}(z^{-1},k) - \hat{L}(z^{-1},j+3r-1)\}\ u(k)| +$$

$$+ |\gamma(z^{-1})\ B(z^{-1})\ \{\hat{P}(z^{-1},k) - \hat{P}(z^{-1},j+3r-1)\}\ y(k)| +$$

$$+ |\gamma(z^{-1})\ \hat{L}(z^{-1},j+3r-1)\ n(k)| <$$

$$< r(r+1) \cdot \max_{i\varepsilon[1,4-1]} \{1,\ |\hat{l}_i(j+3r-1)|\} \cdot \max_{i\varepsilon[1,r]}$$

$$\{1,|a_i|\} \cdot C + r^2 \cdot \max_{i\varepsilon[0,r-1]} \{|\hat{P}_i(j+3r-1)|\} \cdot$$

$$\max_{i\varepsilon[1,r]} \{|b_i|\} \cdot C + 2r \cdot \max_{i\varepsilon[1,2r+1]} \{|\gamma_i|\} \cdot$$

$$(r-1) \cdot \max_{i\varepsilon[1,r-1]} \{ |\hat{l}_i(k) - \hat{l}_i(j+3r-1)| \} \cdot$$

$$\max_{k\varepsilon[j-r,j+s]} \{|u(k)|\} + 2r \cdot \max_{i\varepsilon[0,2r-1]} \{|\gamma_i|\} \cdot r \cdot$$

$$\max_{i\varepsilon[1,r]} \{|b_i|\} \cdot r \cdot \max_{i\varepsilon[0,r-1]}\{|\hat{P}_i(k) - \hat{P}_i(j+3r-1)|\} \cdot$$

$$\max_{i\varepsilon[j-r,j+s+1]} \{|y(k)|\} + 2r \cdot \max_{i\varepsilon[0,2r-1]} \{|\gamma_i|\} \cdot$$

$$r \cdot \max_{i\varepsilon[1,r-1]} \{1, |\hat{l}_i(j+3r-1)|\} \cdot \Delta_b$$

or

$$|\gamma(z^{-1})\; B(z^{-1})\; \hat{P}(z^{-1},k)\; y^*(k)| < \underline{\Omega} \cdot C + \xi\, M\, ||\gamma||\, (\Delta\theta)_{max} +$$

$$2r \cdot r \cdot \max_{i\varepsilon[0,2r-1]} \{|\gamma_i|\} \cdot \max_{i\varepsilon[1,r-1]} \{1, |\hat{l}_i(j+3r-1)|\} \cdot \Delta_b \qquad (3.61)$$

where

$$\underline{\Omega} = r(r+1) \cdot \max_{i\varepsilon[1,r-1]} \{1, |\hat{l}_i(j+3r-1)|\} \cdot \max_{i\varepsilon[1,r-1]} \{1, |a_i|\} +$$

$$r^2 \cdot \max_{i\varepsilon[0,r-1]} \{|\hat{P}_i(j+3r-1)|\} \cdot \max_{i\varepsilon[1,r]} \{|b_i|\}$$

$$M = \max \;\; \max_{k\varepsilon[j-r,j+s]}\{|u(k)|\} \;,\; \max_{k\varepsilon[j-r,j+s]}\{|y(k)|\}$$

$$(\Delta\theta)_{max} = \max \{[\; |\hat{l}_i(k) - \hat{l}_i(j+3r-1)|],\; |\hat{P}_i(k) - \hat{P}_i(j+3r-1)|\}$$

when $i = 0, 1, \ldots, r-1$ and $k = j, j+1, \ldots, j+s+1$

$$\xi = (2r+1) \cdot r^2$$

If we define

$$\delta(z^{-1}) = \gamma(z^{-1})\, B(z^{-1})\, \hat{P}(z^{-1},k) \qquad (3.61a)$$

we shall have

$$\begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{4r-2} \end{bmatrix} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} < \underline{\Omega} \cdot C + \xi\, M\, ||\gamma||\, (\Delta\theta)_{max} + 2r \cdot r \cdot \max_{i\varepsilon[0,2r-1]} \{|\gamma_i|\} \cdot \max_{i\varepsilon[1,r-1]} \{1, |\hat{l}_i(j+3r-1)|\} \cdot \Delta_b$$

for all k ϵ[j+3r-1, j+s+1] which proves lemma 3.4.

Lemma 3.5

Suppose that the system is described by equation (3.11) and $A(z^{-1})$ and $B(z^{-1})$ are prime polynomials. Also, suppose that thete exists $\epsilon_0 \ldots \epsilon_{r-1}$, $\pi_0 \ldots \pi_{r-1}$, not all zero such that

$$\begin{bmatrix} \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{r-1} & \pi_0 & \pi_1 & \cdots & \pi_{r-1} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-r+1) \\ u(k) \\ u(k-1) \\ \vdots \\ u(k-r+1) \end{bmatrix} < C \qquad (3.62)$$

for all k $\epsilon[j, j+s]$.

Then, there exists $\mu_0 \ldots \mu_{2r-1}$, not all zero, such that

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{2r-1} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-2r+1) \end{bmatrix} < r \cdot \max_{i\varepsilon[1,r]}\{|b_i|\} \cdot C + \max_{i\varepsilon[1,r-1]}\{|\pi_i|\} \cdot \Delta_b$$

for all k $\epsilon[j+r, j+s+1]$

Proof

Inequality (3.62) means that

$$|\varepsilon(z^{-1})\ y(k) + \pi(z^{-1})\ u(k)| < C$$

for all k $\epsilon[j, j+s]$

This implies that for all k $\epsilon[j+r, j+s+1]$

$$|B(z^{-1})\ \varepsilon(z^{-1})\ y(k) + B(z^{-1})\ \pi(z^{-1})\ u(k)| = |[B(z^{-1})\ \varepsilon(z^{-1}) +$$

$$A(z^{-1})\ \pi(z^{-1})]y(k) - \pi(z^{-1})n(k)|$$

But

$$|B(z^{-1})\ [\varepsilon(z^{-1})y(k) + \pi(z^{-1})u(K)]| < r \cdot \max_{i\varepsilon[1,r]}\{|b_i|\} \cdot C$$

We define $\mu(z^{-1})$ such that

$$\mu(z^{-1}) = B(z^{-1})\ \varepsilon(z^{-1}) + A(z^{-1})\ \pi(z^{-1})$$

then

$$|\mu(z^{-1})y(k)| < r \cdot \max_{i\varepsilon[1,r]} \{|b_i|\} \cdot + \max_{i\varepsilon[1,r-1]} \{|\pi_i|\} \cdot \Delta_b \qquad (3.64)$$

We have $\mu(z^{-1}) \neq 0$ since $B(z^{-1})$ and $A(z^{-1})$ are prime. Since the degree of $\mu(z^{-1})$ is $2r-1$, inequality (3.64) can be written as

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{2r-1} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-2r+1) \end{bmatrix} < r \cdot \max_{i\varepsilon[1,r]} \{|b_i|\} \cdot C + \max_{i\varepsilon[1,r-1]} \{|\pi_i|\} \cdot \Delta_b$$

for all $k \epsilon [j+r, j+s+1]$ and this completes the proof.

Theorem 3.2

Suppose that the plant is described by equation (3.11) and that the polynomials $A(z^{-1})$ and $B(z^{-1})$, which are in general of degree r, are prime. The control law is given by equation (3.19) and $\{u(k)\}$ and $\{y(k)\}$ are bounded sequences. Also, suppose that

$$0 < K_3 \, \underline{I} \le \sum_{k=j+3r-1}^{j+s+1} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} \begin{bmatrix} y^*(k) & y^*(k-1) \ldots y^*(k-4r+2) \end{bmatrix} \le K_2 \, \underline{I} < \infty \qquad (3.65)$$

Then

$$0 < \sigma_1 \, \underline{I} \le \sum_{k=j}^{j+s} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-r+1) \\ u(k) \\ u(k-1) \\ \vdots \\ u(k-r+1) \end{bmatrix} \begin{bmatrix} y(k) \; y(k-1) \ldots y(k-r+1) \\ u(k) \; u(k-1) \ldots u(k-r-1) \end{bmatrix} \le \sigma_2 \, \underline{I} < \infty \qquad (3.66)$$

where

$$\sqrt{\sigma_1} < \left[\frac{\eta \cdot \sqrt{K_3}}{\sqrt{s-3r+3}} - \xi \, M \, ||\mu|| \, (\Delta\theta)_{\max} - \nu \right] \cdot \frac{1}{r \cdot \underline{\Omega} \cdot \max\limits_{i \varepsilon [1,r]} \{|b_i|\} \cdot \left\| \begin{bmatrix} \underline{\varepsilon} \\ \underline{\pi} \end{bmatrix} \right\|} \qquad (3.67)$$

and

$$\nu = (2r-1)(r-1) \cdot \max_{i \varepsilon [0.2r-1]} \{|\mu_i|\} \cdot \max_{i \varepsilon [1,r-1]} \{1, |\hat{l}_i(j+3r-1)|\}$$

$$\cdot \Delta_b + \underline{\Omega} \cdot \max_{i \varepsilon [1,r-1]} \{|\pi_i|\} \cdot \Delta_b \qquad (3.68)$$

All the other parameters are as they were defined in lemma 3.4 while η is the norm of $\delta(z^{-1})$.

Proof

The proof of the upper bound of inequality (3.66) follows from the boundness of the sequences $\{u(k)\}$ and $\{y(k)\}$. So, we only have to prove the existence of the lower limit. We shall do it by contradiction.

Suppose that the lower bound fails. So, there exists ϵ_0 ... ϵ_{r-1}, π_0 ... π_{r-1} ,not all zero, such that

$$\begin{bmatrix} \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{r-1} & \pi_0 & \pi_1 & \cdots & \pi_{r-1} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-r+1) \\ u(k) \\ u(k-1) \\ \vdots \\ u(k-r+1) \end{bmatrix} < \sqrt{\sigma_1} \cdot \left\| \begin{bmatrix} \underline{\varepsilon} \\ \underline{\pi} \end{bmatrix} \right\| \qquad (3.69)$$

for all $k \in [j, j+s]$

By lemma 3.5, then, there exists $\mu_0 \ldots \mu_{2r-1}$, not all zero, such that

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{2r-1} \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-2r+1) \end{bmatrix} < r \cdot \max_{i \in [1,r]} \{|b_i|\} \cdot \sqrt{\sigma_1} \cdot \left\| \begin{bmatrix} \underline{\varepsilon} \\ \underline{\pi} \end{bmatrix} \right\| + \max_{i \in [1,r-1]} \{|\pi_i|\} \cdot \Delta_b$$

for all $k \in [j+r, j+s+1]$ and

$$\mu(z^{-1}) = B(z^{-1})\, \varepsilon(z^{-1}) + A(z^{-1})\, \pi(z^{-1})$$

By lemma 3.4, there exists $\delta_0 \ldots \delta_{4 -2}$, not all zero, such that

$$\begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{4r-2} \end{bmatrix} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} < \underline{\Omega} \cdot \left\{ r \cdot \max_{i \in [1,r]} \{|b_i|\} \cdot \sqrt{\sigma_1} \cdot \left\| \begin{bmatrix} \underline{\varepsilon} \\ \underline{n} \end{bmatrix} \right\| + \max \{|\pi_i|\} \right\} \cdot \Delta_b$$

$$+ \xi \, M \, \|\mu\| \, (\Delta\theta)_{max} + (2r-1)(r-1) \cdot \max_{i \in [0,2r-1]} \{|\mu_i|\} \cdot \max_{i \in [1,r-1]} \{l_1, |\hat{l}_i(j+3r-1)|\} \, \Delta_b \tag{3.70}$$

If we have

$$\sqrt{\sigma_1} < \left[\frac{\eta \sqrt{K_3}}{\sqrt{s-3r+3}} - \xi \; M \; ||\mu|| \; (\Delta\theta)_{\max} - \nu \right] \cdot \frac{1}{r \cdot \underline{\Omega} \cdot \max \; \{|b_i|\} \cdot \left\| \begin{bmatrix} \underline{\varepsilon} \\ \underline{\pi} \end{bmatrix} \right\|} \qquad (3.71)$$

$$\nu = 2r \cdot r \cdot \max_{i\varepsilon[1,24-1]} \{|\mu_i|\} \cdot \max_{i\varepsilon[1,r-1]} \{1, |\hat{1}_i(j+3r-1)|\} \cdot$$

$$\Delta_b + \underline{\Omega} \cdot \max_{i\varepsilon[1,r-1]} \{|\pi_i|\} \cdot \Delta_b \qquad (3.72)$$

$$\eta = ({\delta_0}^2 + {\delta_1}^2 + \dots + \delta_{4r-2}^2)^{1/2}$$

then inequalities (3.70) and (3.71) give

$$\begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{4r-2} \end{bmatrix} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} < \frac{\eta \quad \sqrt{K_3}}{\sqrt{s-3r+3}} \qquad (3.72)$$

Inequality (3.72) gives

$$\underline{\delta}^T \cdot \sum_{k=j+3r-1}^{j+s+1} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} \begin{bmatrix} y^*(k) & y^*(k-1) \ldots y^*(k-4r+2) \end{bmatrix} \cdot \underline{\delta} < \eta^2 \cdot K_3$$

which means that

$$\sum_{k=j+3r-1}^{j+s+1} \begin{bmatrix} y^*(k) \\ y^*(k-1) \\ \vdots \\ y^*(k-4r+2) \end{bmatrix} \begin{bmatrix} y^*(k) & y^*(k-1) \ldots y^*(k-4r+2) \end{bmatrix} < K_3 \cdot \underline{I} \tag{3.73}$$

since $\eta^2 = \underline{\delta}' \, \underline{\delta}$

Inequality (3.73) contradicts(3.65) and the theorem is proved.

Remark 1

The control algorithm (3.11) to (3.26) cannot guarantee that the output will track the reference signal $y^*(k)$. If we require minimization of the tracking error, we must include an integrator. In this case, equations (3.18) and (3.19) will be substituted by the following three equations

$$(1-z^{-1})\ \hat{A}(z^{-1},k)\ \hat{L}(z^{-1},k) + \hat{B}(z^{-1},k)\ \hat{P}(z^{-1},k) = A^{*}(z^{-1}) \tag{3.74}$$

$$\hat{L}(z^{-1},k)\ v(k) = \hat{P}(z^{-1},k)\ [y^{*}(k) - y(k)] \tag{3.75}$$

$$(1-z^{-1})\ u(k) = v(k) \tag{3.76}$$

where $v(k)$ is an auxiliary signal while the polynomials $\hat{L}(z^{-1},k)$ and $\hat{P}(z^{-1},k)$ are both of order $r-1$ where r is defined as

$$r = \max\ \{n+1,\ m+d\}$$

Because of the integrator, equation (3.74) will have a unique solution if $B(z^{-1})$ does not have roots on the unit circle. The control configuration can be seen in figure 3.4. From figure 3.4 we can find that

$$y(k) = \frac{P(z^{-1})B(z^{-1})}{(1-z^{-1})A(z^{-1})L(z^{-1}) + B(z^{-1})P(z^{-1})}\ y^{*}(k) + \frac{(1-z^{-1})A(z^{-1})L(z^{-1})}{(1-z^{-1})A(z^{-1})L(z^{-1}) + B(z^{-1})P(z^{-1})}\ \eta(k) \tag{3.77}$$

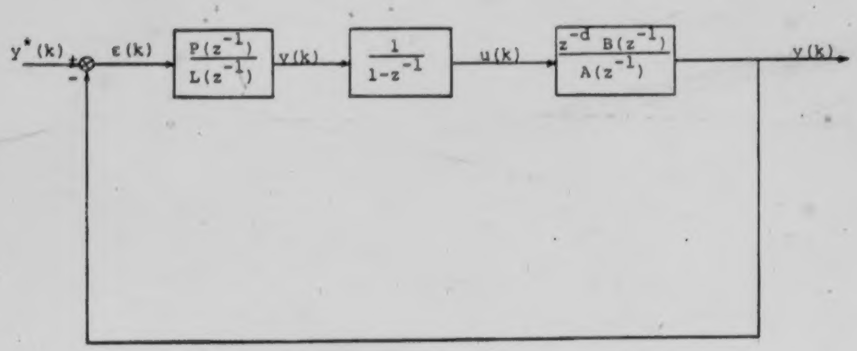


Figure 3.4. Control scheme with integral action.

After steady-state conditions have been established, equation (3.77) reduces to the following

$$y(k) = y^{*}(k) + \frac{A(z^{-1})\ L(z^{-1})}{B(z^{-1})\ P(z^{-1})}\,[n(k) - n(k-1)] \qquad (3.78)$$

If the tracking error $\epsilon(k)$ is defined as

$\epsilon(k) = y^{*}(k) - y(k)$

equation (3.78) gives that

$$|\varepsilon(k)| < 2\left[\frac{\Sigma a_i\ \ \Sigma l_i}{\Sigma b_i\ \ \Sigma \rho_i}\right] \cdot \Delta_b \qquad (3.79)$$

Inequality (3.79) means that the tracking error will be bounded, for the stochastic case. In the deterministic case, the tracking error $\epsilon(k)$ will converge to zero

.

3.5 Simulation results

In this section, a number of simulation results will be presented, illustrating the performance of the presently discussed adaptive pole-assignment algorithms. Since in these simulation results we are concerned about the tracking problem, the pole-assignment algorithm with integral action has been used. We have considered five different models (minimum or nonminimum phase) for the deterministic without time delay, the deterministic with time delay and stochastic cases.

The first simulation example studies the model

$$G(z^{-1}) = \frac{0.2\, z^{-1} + 0.12\, z^{-2}}{1 - 1.2\, z^{-1} + 0.52\, z^{-2}}$$

which is a minimum phase system. The second simulation example has also been considered in Cameron and Seborg (1983) and is the following minimum phase system

$$G(z^{-1}) = \frac{0.4809\, z^{-1} + 0.2725\, z^{-2}}{1 - 1.0382\, z^{-1} + 0.2466\, z^{-2}}$$

The third simulation example is the nonminimum phase system

$$G(z^{-1}) = \frac{0.5\ z^{-1} + 1.\ z^{-2}}{1 - 2\ z^{-1} + 0.99\ z^{-2}}$$

and has also been considered in Goodwin and Sin (1981). The fourth and fifth simulation examples are nonminimum phase systems and have also been considered in Clarke (1984). These are the following systems

$$G(z^{-1}) = \frac{0.1\ z^{-1} + 0.2\ z^{-2}}{1 - 0.7\ z^{-1}}$$

and

$$G(z^{-1}) = \frac{0.1\ z^{-1} + 0.2\ z^{-2}}{1 - 0.7\ z^{-1} + 0.72\ z^{-2}}$$

respectively.

In all the simulation runs, regardless of the system order, the desired closed loop characteristic polynomial has the form $A^*(z^{-1}) = 1 + a_1^*\ z^{-1}$. By considering this form of closed loop polynomial, we can effectively assign only one of the system poles. Also, the initial parameter vector was chosen to be equal to zero. Because of it, we initialy use a fixed gain controller during the time period when the singularity of the Sylvester matrix is limiting the solution of the pole-assignment algorithm.

It was found that the control algorithm given by equations (3.75) and (3.76) leads to unacceptably large inputs and oscillatory system response when a sudden set point change occurs This can be seen in figure 3.5. One way to improve the controller performance is to consider the modified control law.

$$\hat{L}(z^{-1},k)\ v(k) = \hat{P}(z^{-1},k)\ [y^{*} - y(k)] \qquad (3.80)$$

$$(1 - z^{-1})\ u(k) = v(k) \qquad (3.81)$$

By using this control law, the control signal u(k) at the discrete time k only depends on the set point value, y^{*}, at the same time k and not on the set point values at previous time instants. The improvement in the controller performance can be seen by comparing figure 3.5 with figure 3.6. The fast and exact estimation of the system parameters can be seen in figure 3.7. The trajectory of the controller parameters can be seen in figure 3.8. Note that the fixed gain controller has been used only for the first three time instants. This fixed gain controller must be carefully chosen since it affects the system response, especially during the initial transient state (compare figure 3.6 with figure 3.9 and figure 3.10 with figure 3.12). It is very interesting to see that the identification scheme estimates the true system parameters (figure 3.11) despite the large variations of the process input and output (figure 3.10). This is in agreement with the stability analysis of the identification scheme in section 2.3 of the second chapter. Instead of using the control law (3.80) and (3.81), another way of improving the system response is to vary the pole location (compare figure 3.10 with figure 3.13). This method has the disadvantage that the best pole location cannot be easily found. On the other hand, the modified control law (3.68) to (3.69) seems to be insensitive to the pole location.

In figure 3.14 we can see the response of the first nonminimum phase system. The sharp oscillations of the system response during the initial transient phase can be attributed to the initially non-converged system parameters and to the high system gain.

In figure 3.15 we have reproduced the same results as those presented by Clarke (1984) where a self-tuning pole assignment algorithm was considered. The performance of the proposed pole-assignment algorithm, illustrated by this simulation example is better than the performance of the self-tuning pole assignment algorithm presented by Clarke (1984).

Figures 3.16 and 3.17 illustrate the performance of the pole-assignment algorithm trying to control the second nonminimum phase system taken from Clarke (1984). So far, the performance of the pole-assignment algorithm has been shown to be excellent. This is the case when there is no model mismatch. Figure 3.18 illustrates the performance of the pole-assignment algorithm for the fifth simulation example when the denominator of the process was modelled as a first order polynomial. It is very obvious that this control algorithm is very sensitive to model mismatch.

Figures 3.19 to 3.21 have been drawn for three different simulation examples when a time delay of three sampling periods was added to the system model. The performance of the controller can be considerd as excellent despite its deterioration compared to the one (compare with the corresponding figures 3.6, 3.12 and 3.17) of the delay free deterministic case.

Figures 3.22 to 3.25 illustrate the performance of the controller trying to control the first simulation example for the delay free stochastic case. The noise added to the system is a Gaussian noise with zero mean value and variance equal to 0.005. By comparing figures 3.22 and 3.25 we can easily see that the controller performance deteriorates as the level of the noise increases. In figure 3.26 we can see the trajectories of the forgetting factors for two different levels of system noise. In the stochastic case, the forgetting factor does not converge to its upper limit since the prediction error does not converge to zero.

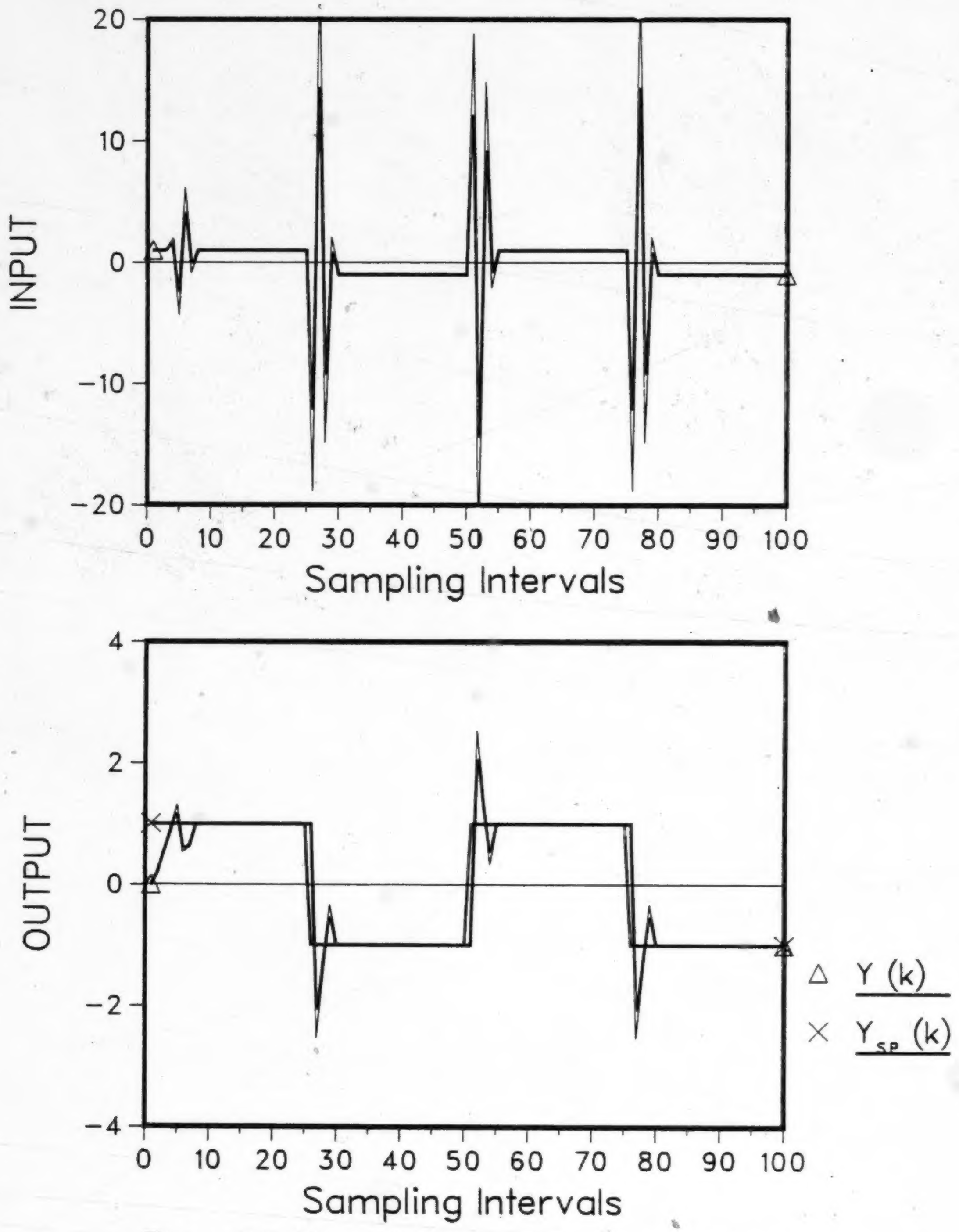


Fig.–3.5 Simulated process response
Ex 1/PA/IA/DC/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

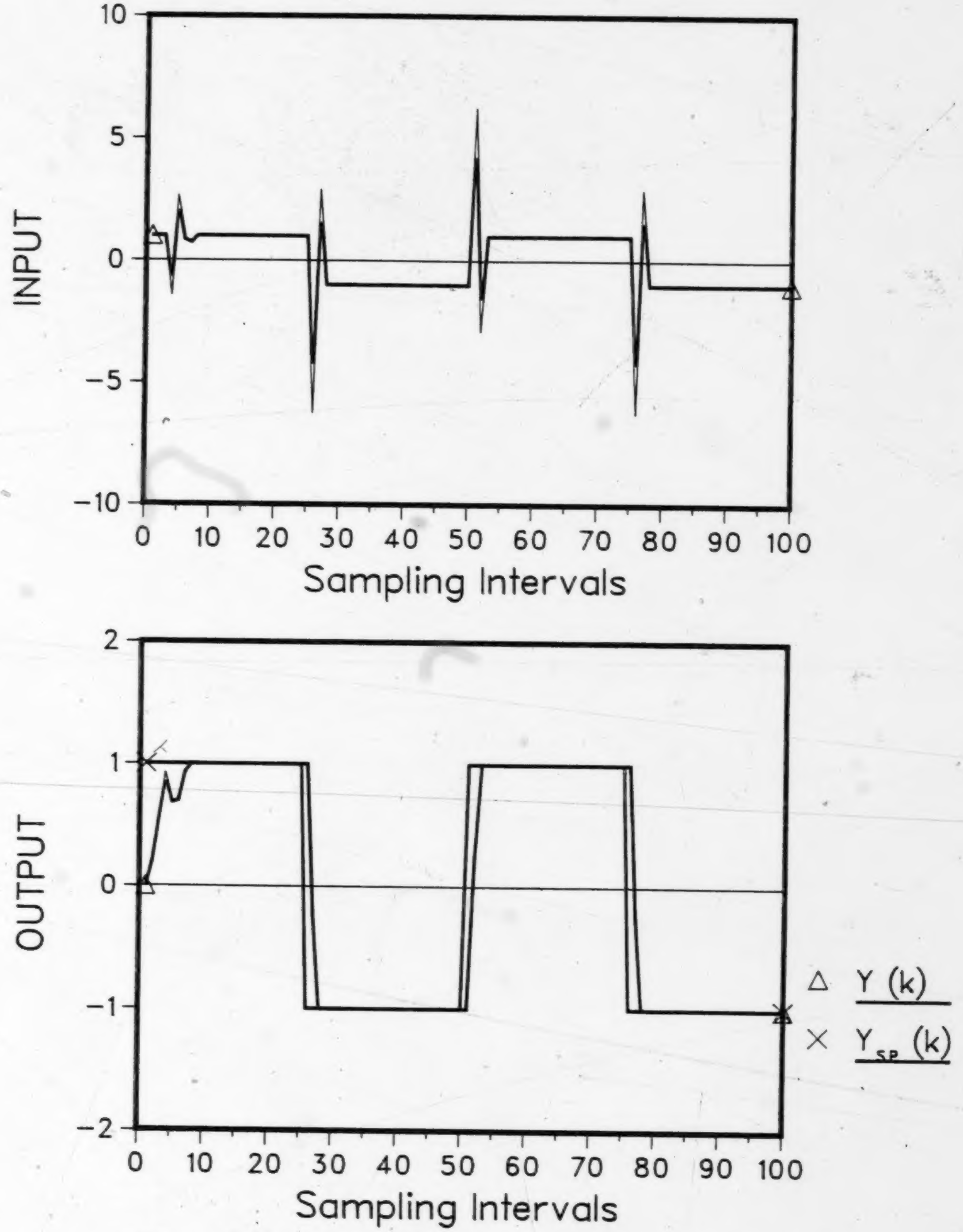
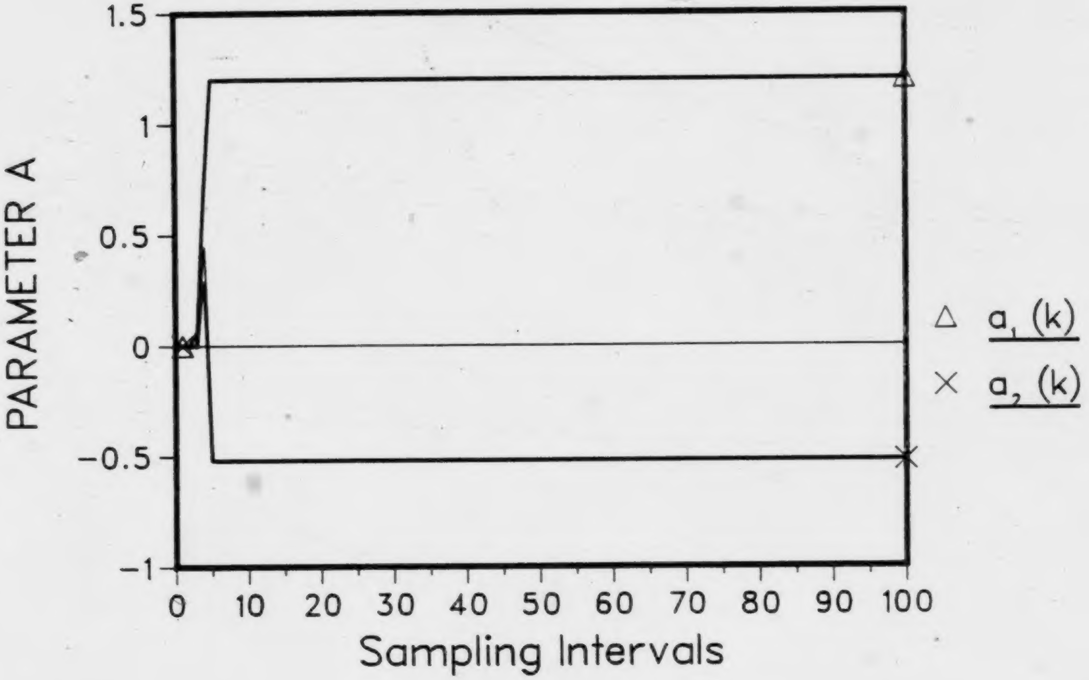


Fig.−3.6 Simulated process response
Ex 1/PA/IA/DC/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98



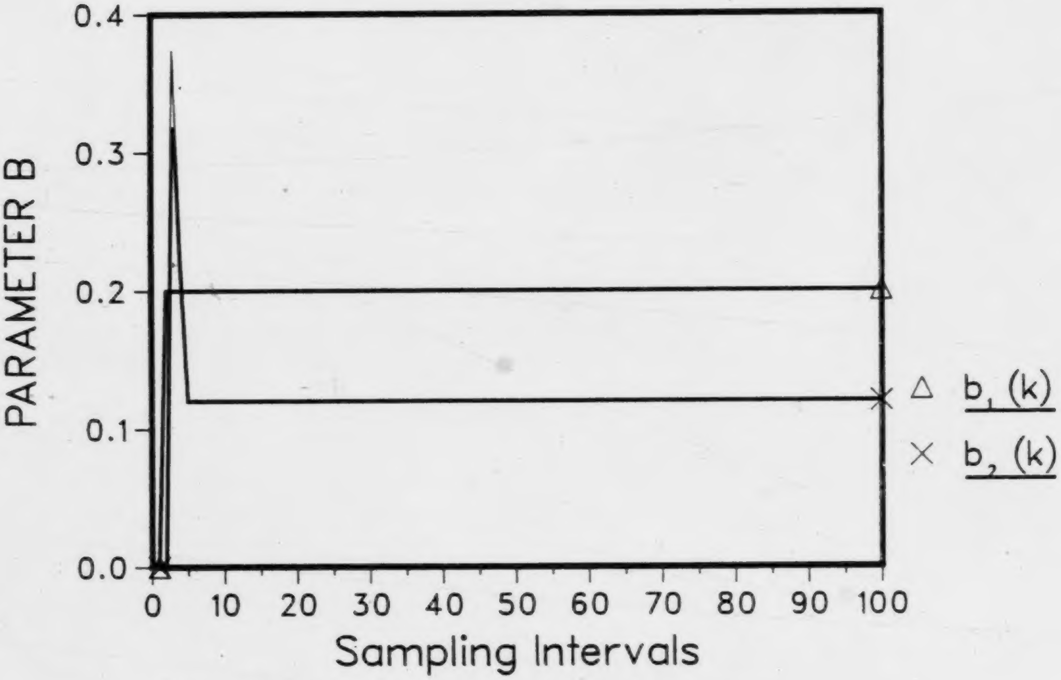


Fig.−3.7 Parameter trajectory
Ex 1/PA/IA/DC/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

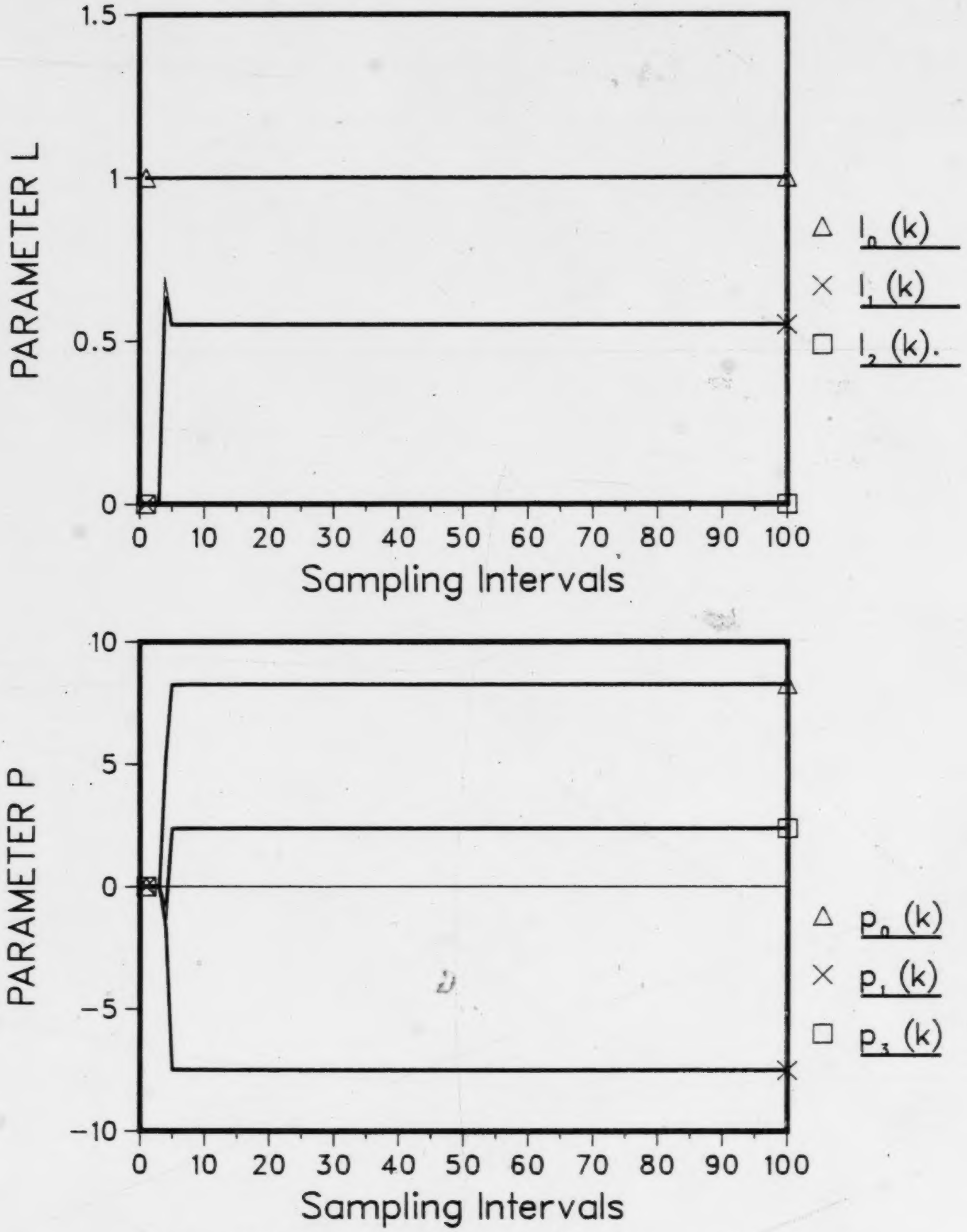


Fig.−3.8 Parameter trajectory of PA
Ex 1/PA/IA/DC/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

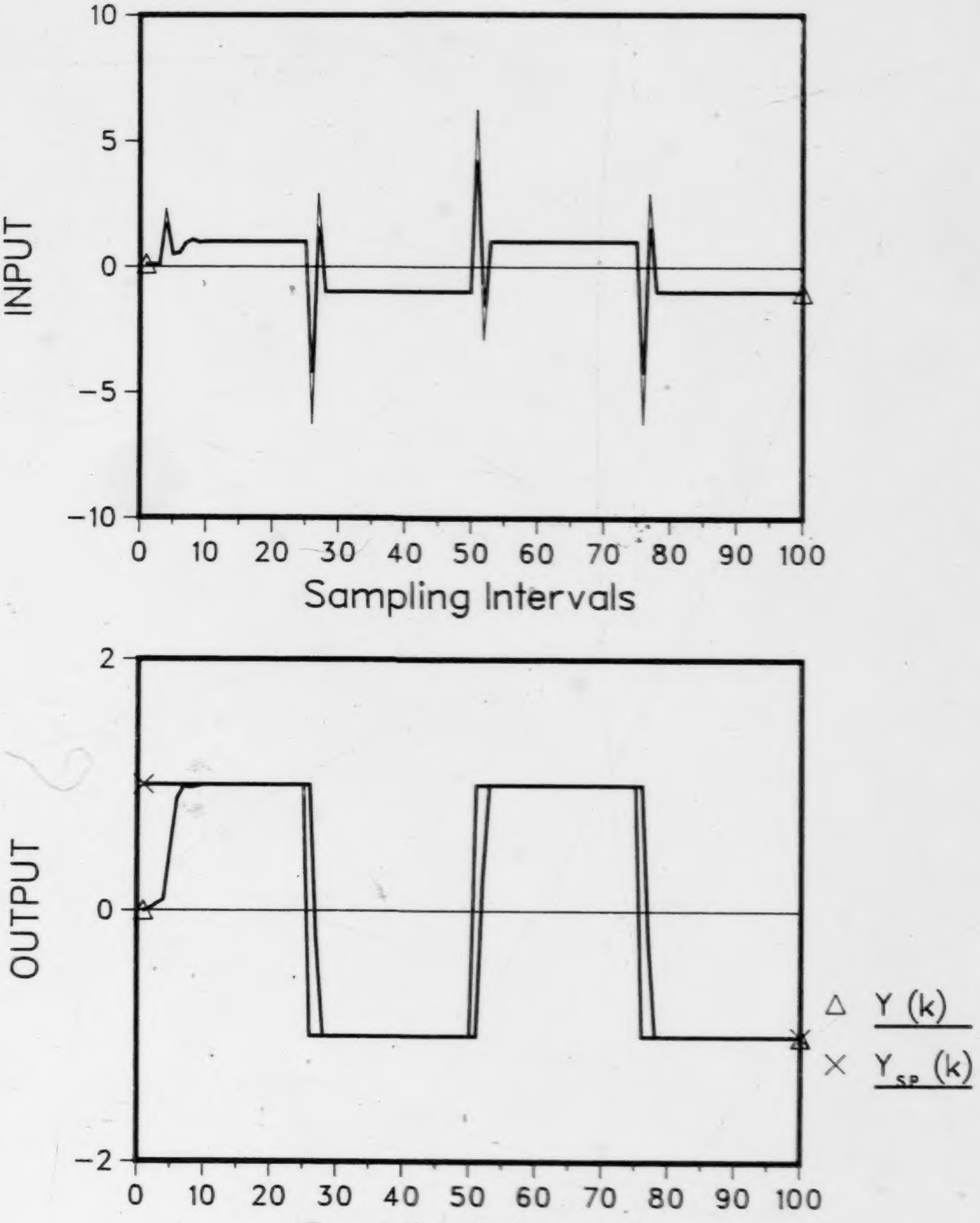


Fig.–3.9 Simulated process response
Ex 1/PA/IA/DC/d=0/p=0./Uf=0.1
N_0=10 P(0)=1.E6 λ_{max}=0.98

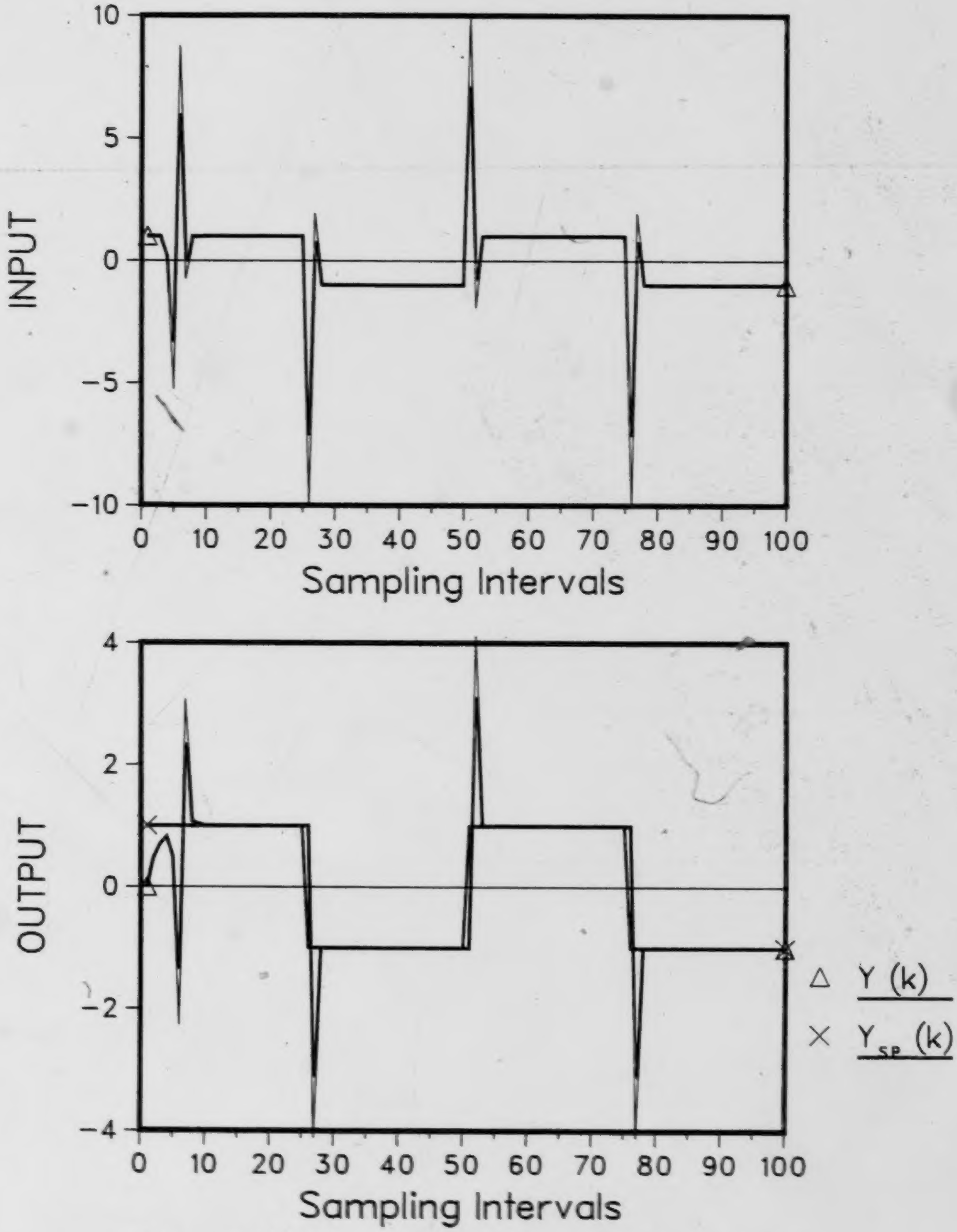


Fig.−3.10 Simulated process response
Ex 2/PA/IA/DC/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

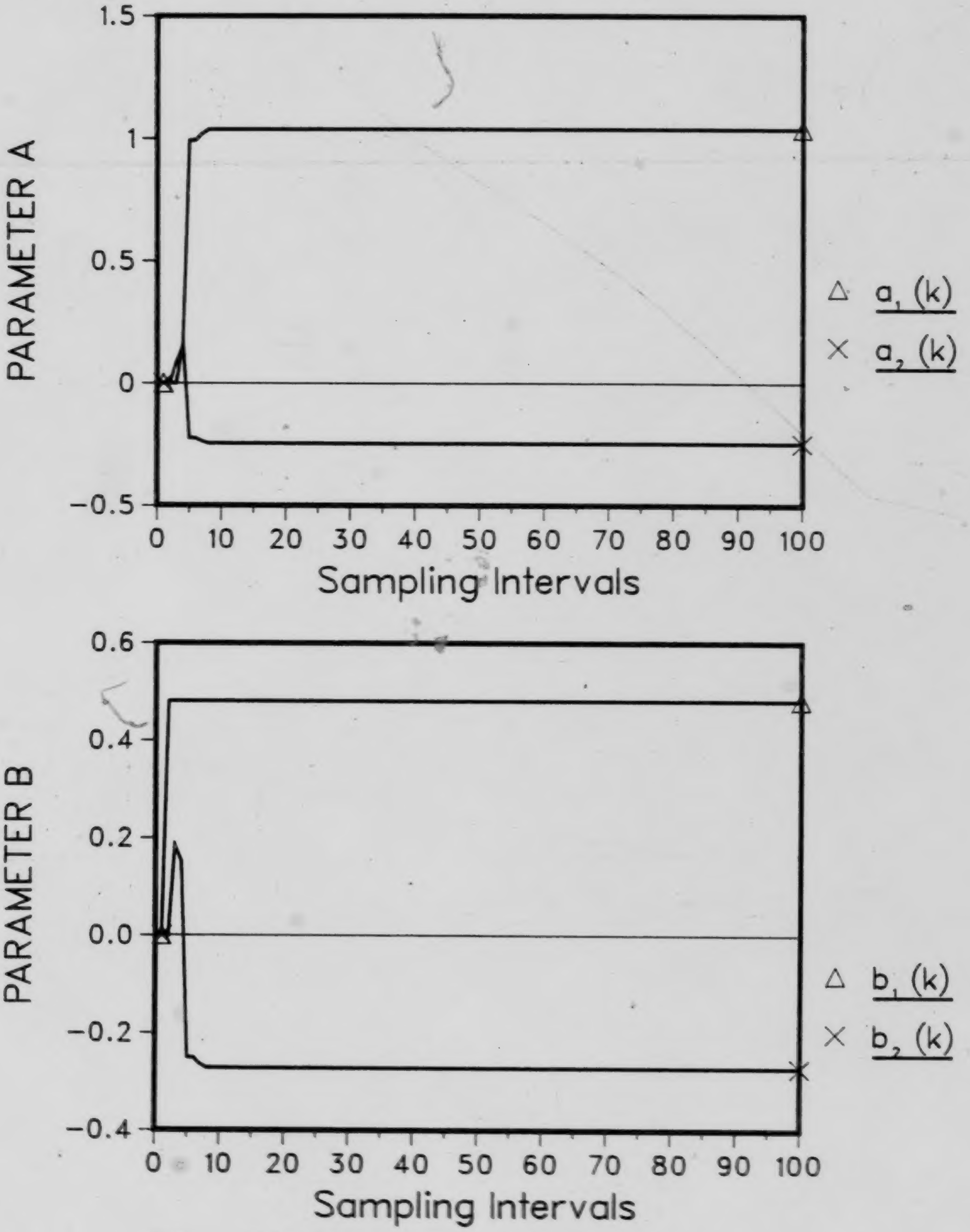


Fig.–3.11 Parameter trajectory
Ex 2/PA/IA/DC/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

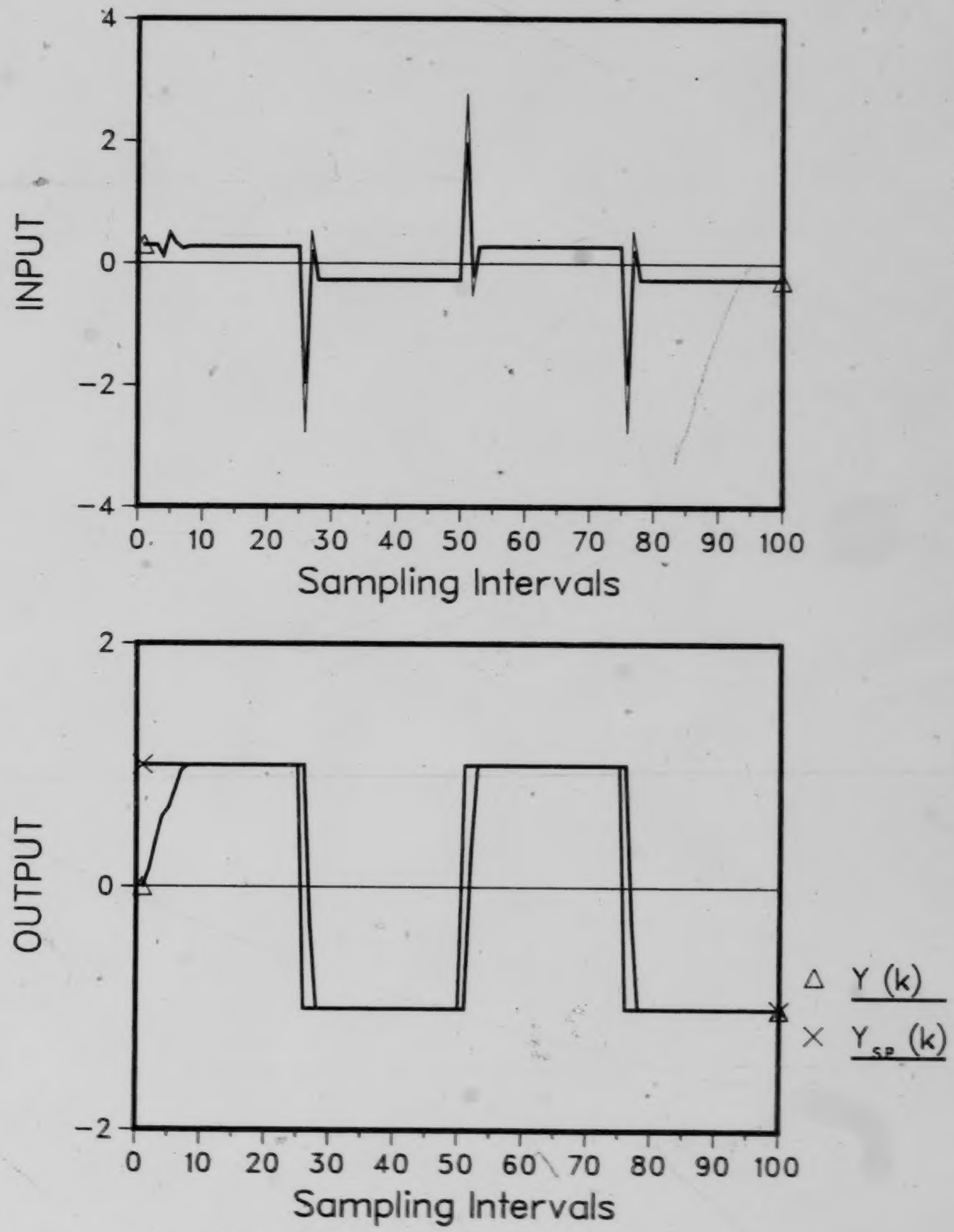


Fig.–3.12 Simulated process response
Ex 2/PA/IA/DC/d=0/p=0./Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

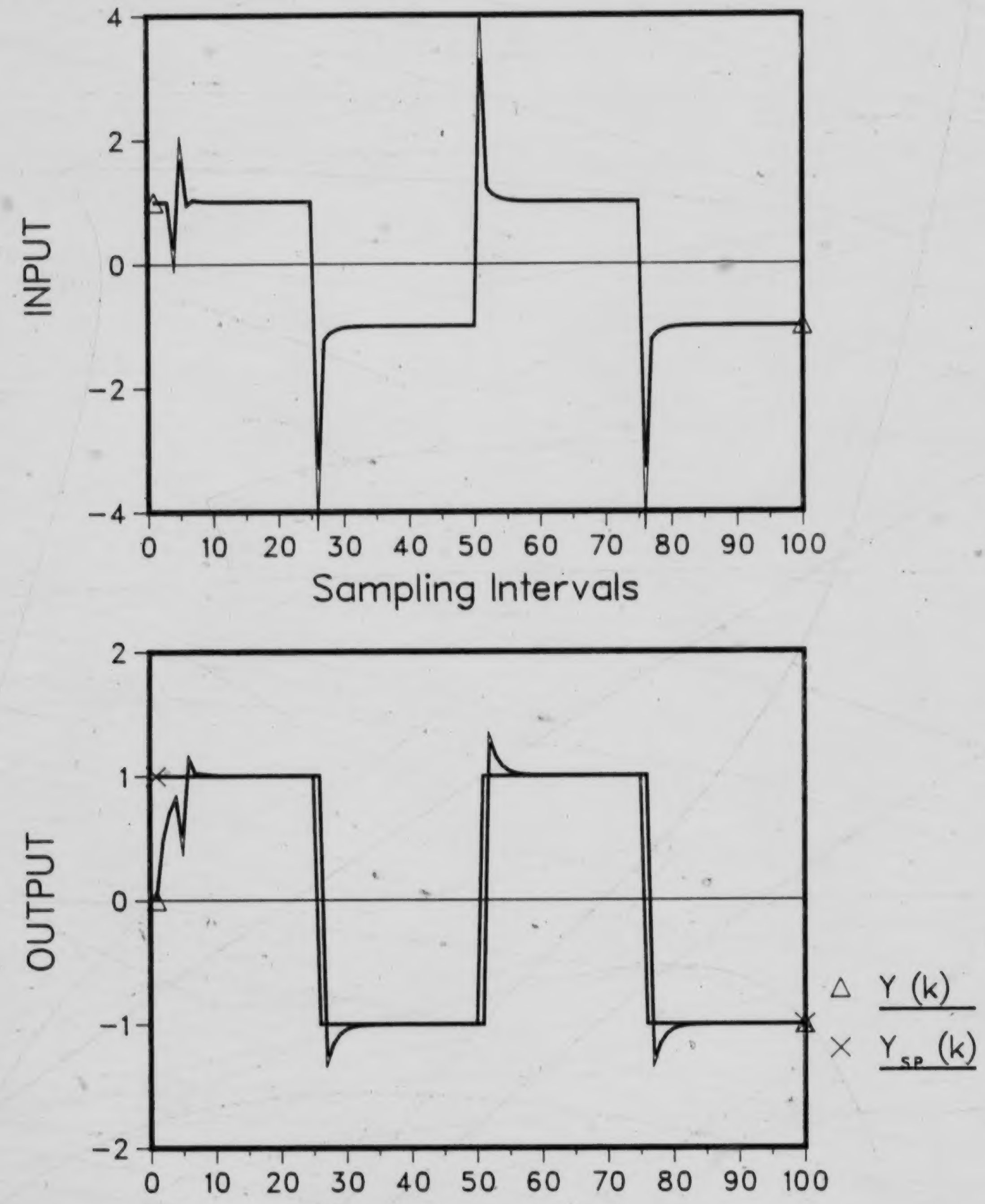


Fig.−3.13 Simulated process response
Ex-2/PA/IA/DC/d=0/p=0.5/Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

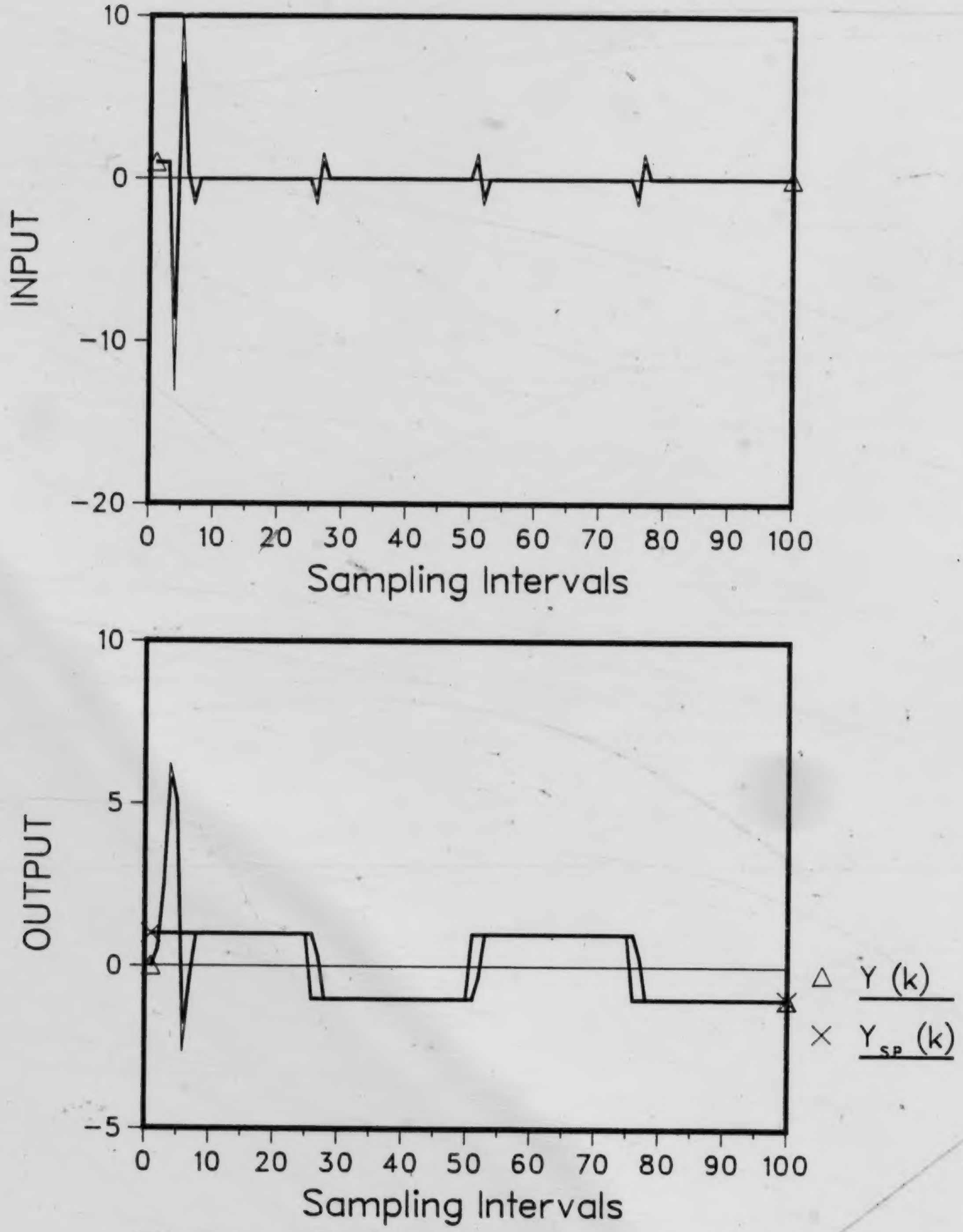


Fig.—3.**14** Simulated process response
Ex 3/PA/IA/DC/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

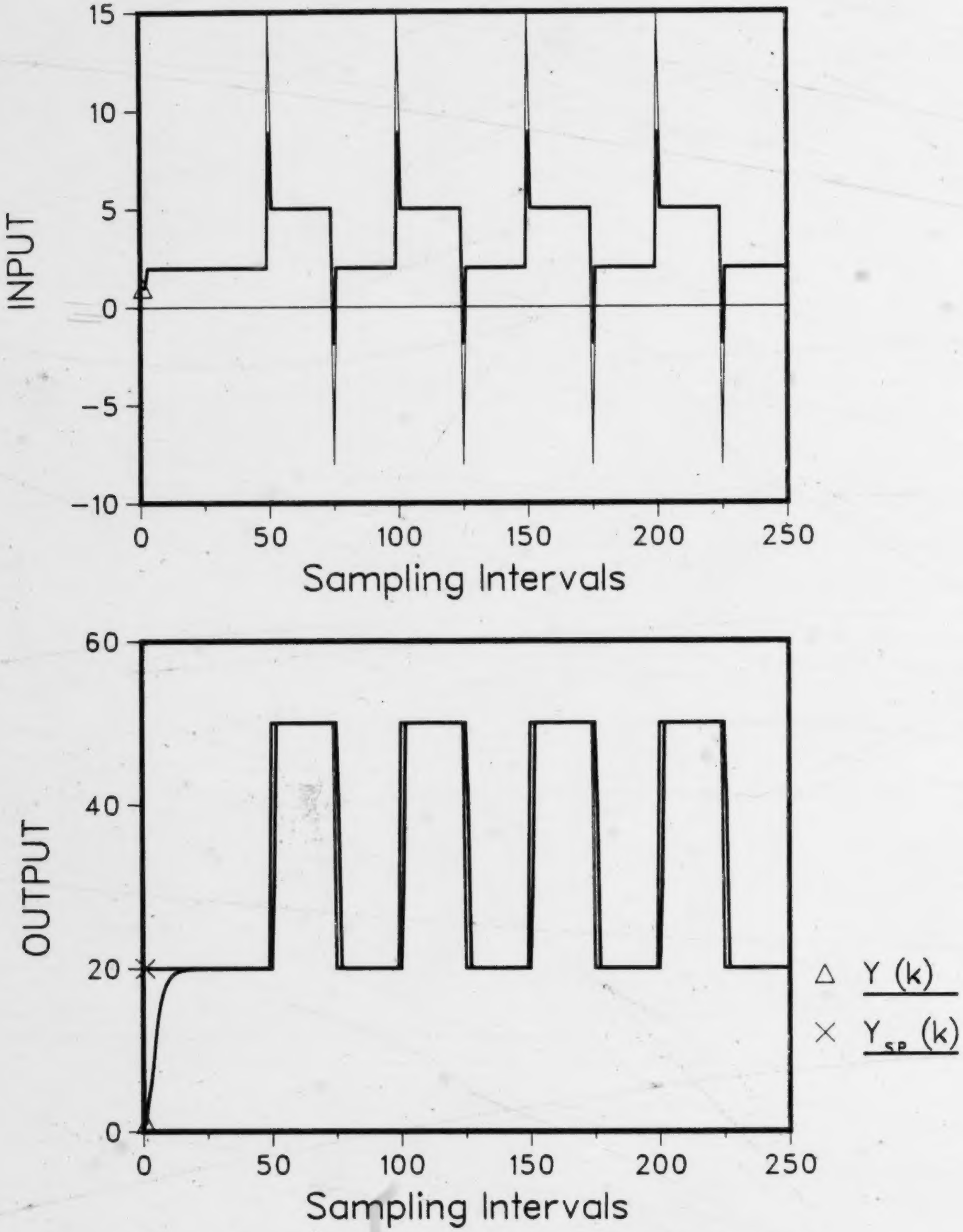


Fig.−3.**15** Simulated process response
Ex 4/PA/IA/DC/d=0/p=0.7/Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

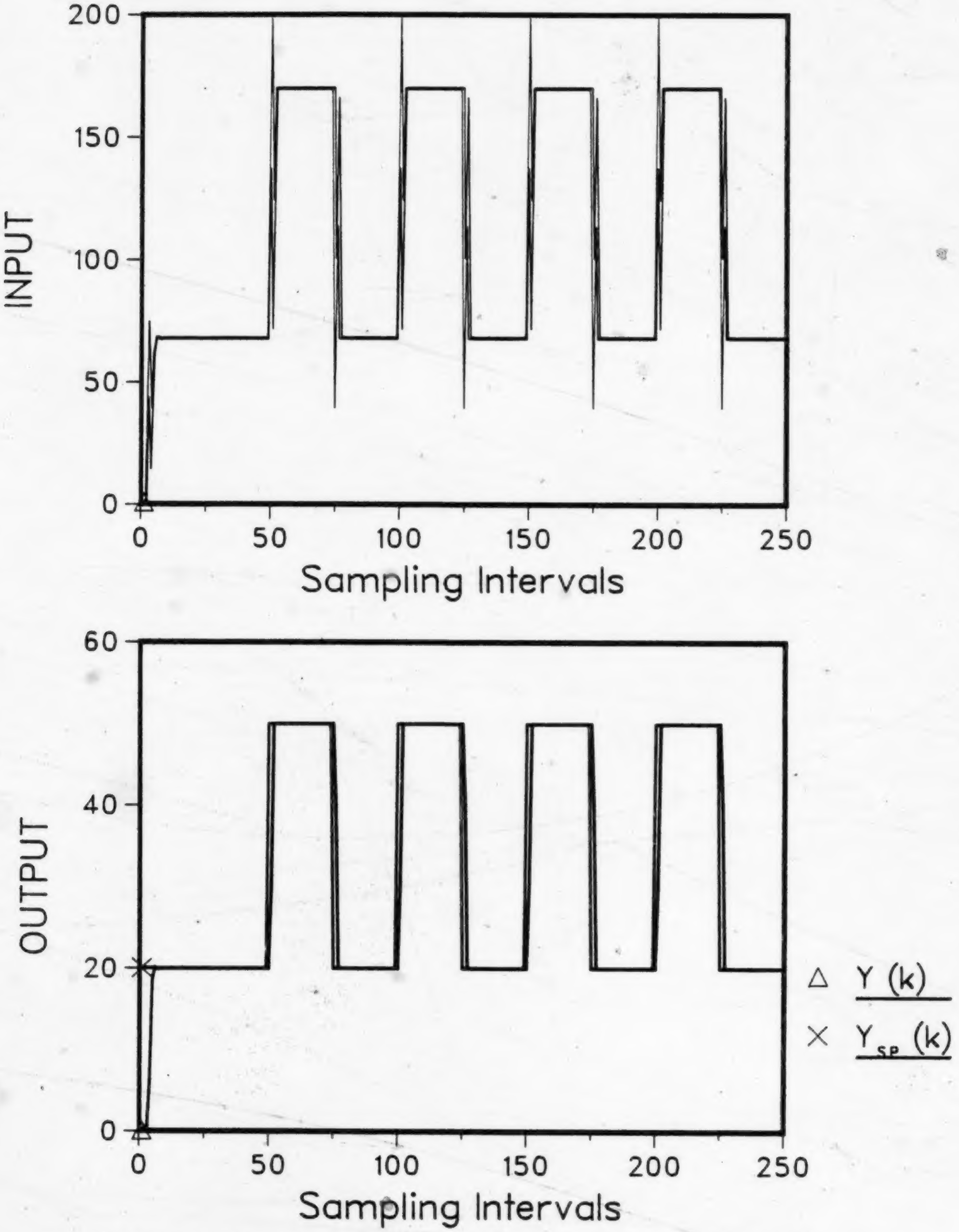


Fig.–3.**16** Simulated process response
Ex 5/PA/IA/DC/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

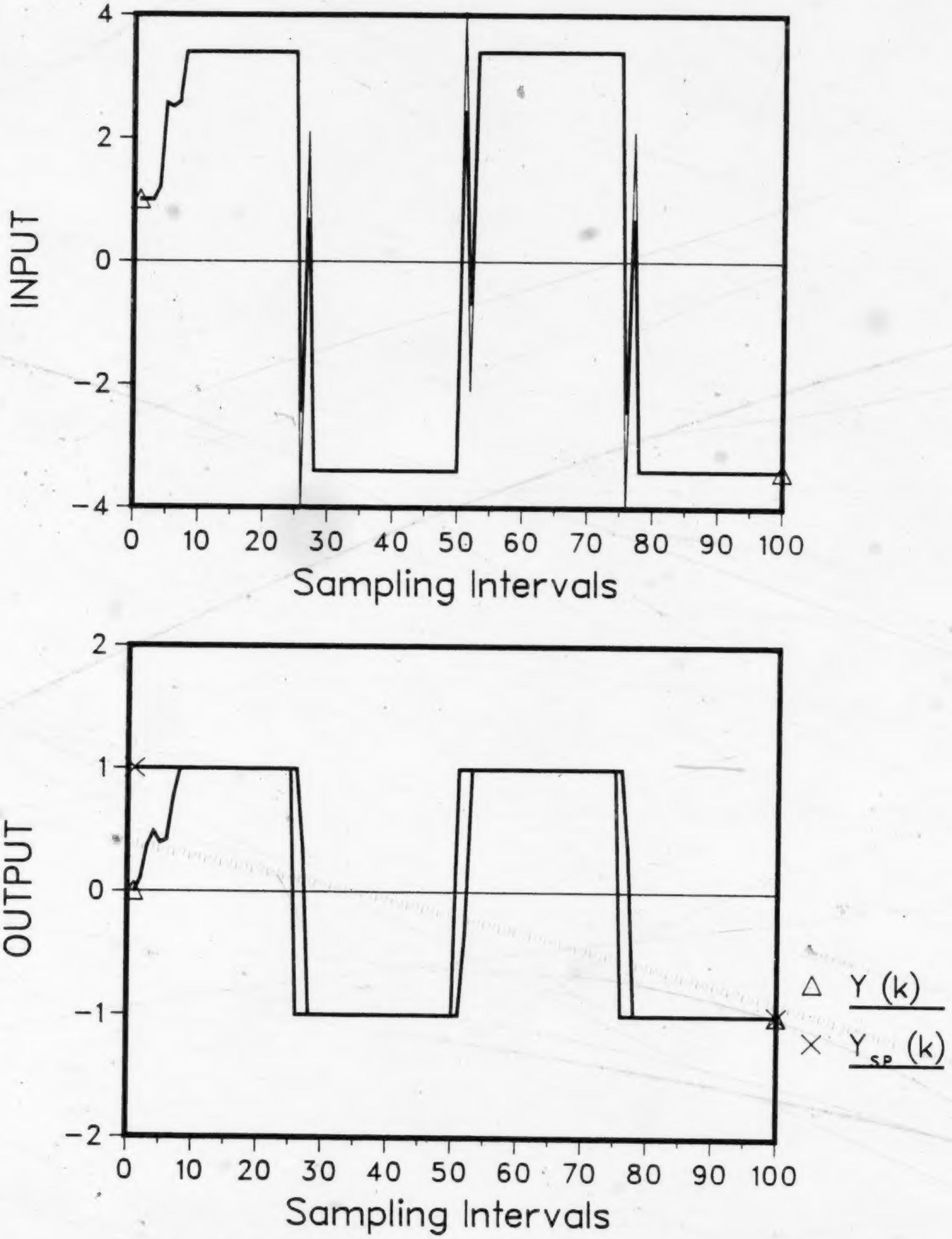


Fig.—3.**17** Simulated process response
Ex 5/PA/IA/DC/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

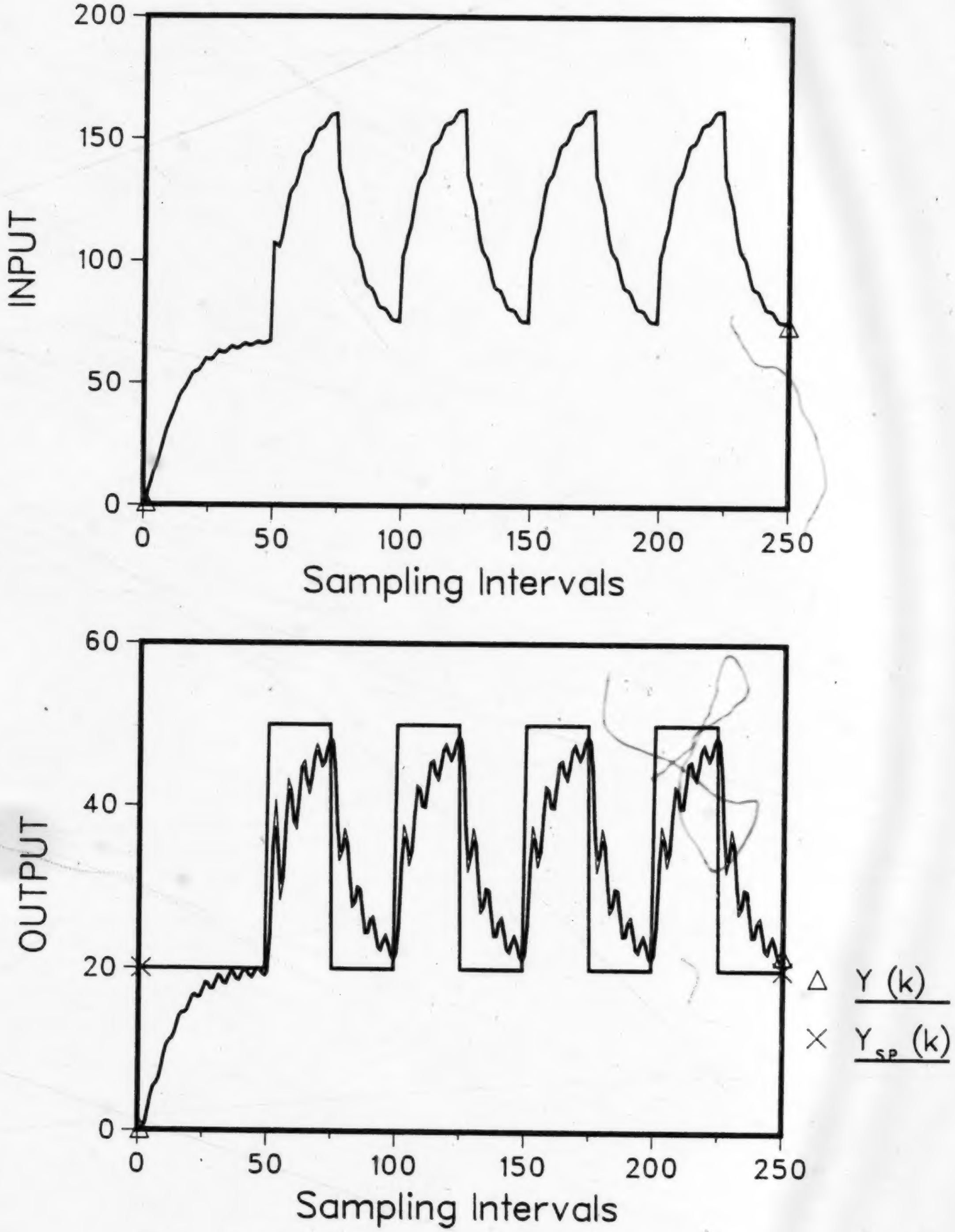


Fig.—3.18 Simulated process response
Ex 5/PA/IA/DC/d=0/p=0.9/Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

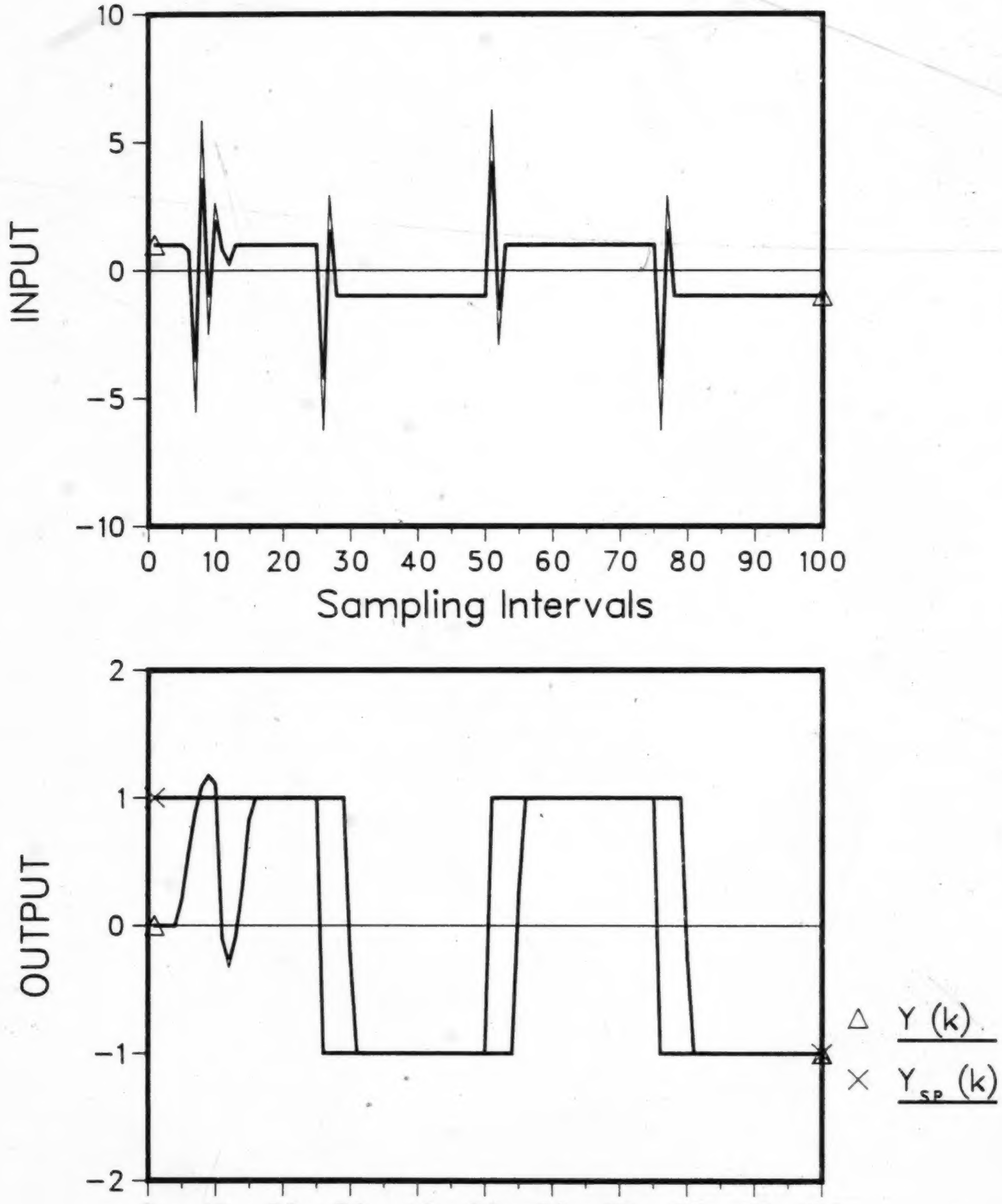


Fig.−3.**19** Simulated process response
Ex 1/PA/IA/DC/d=3/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

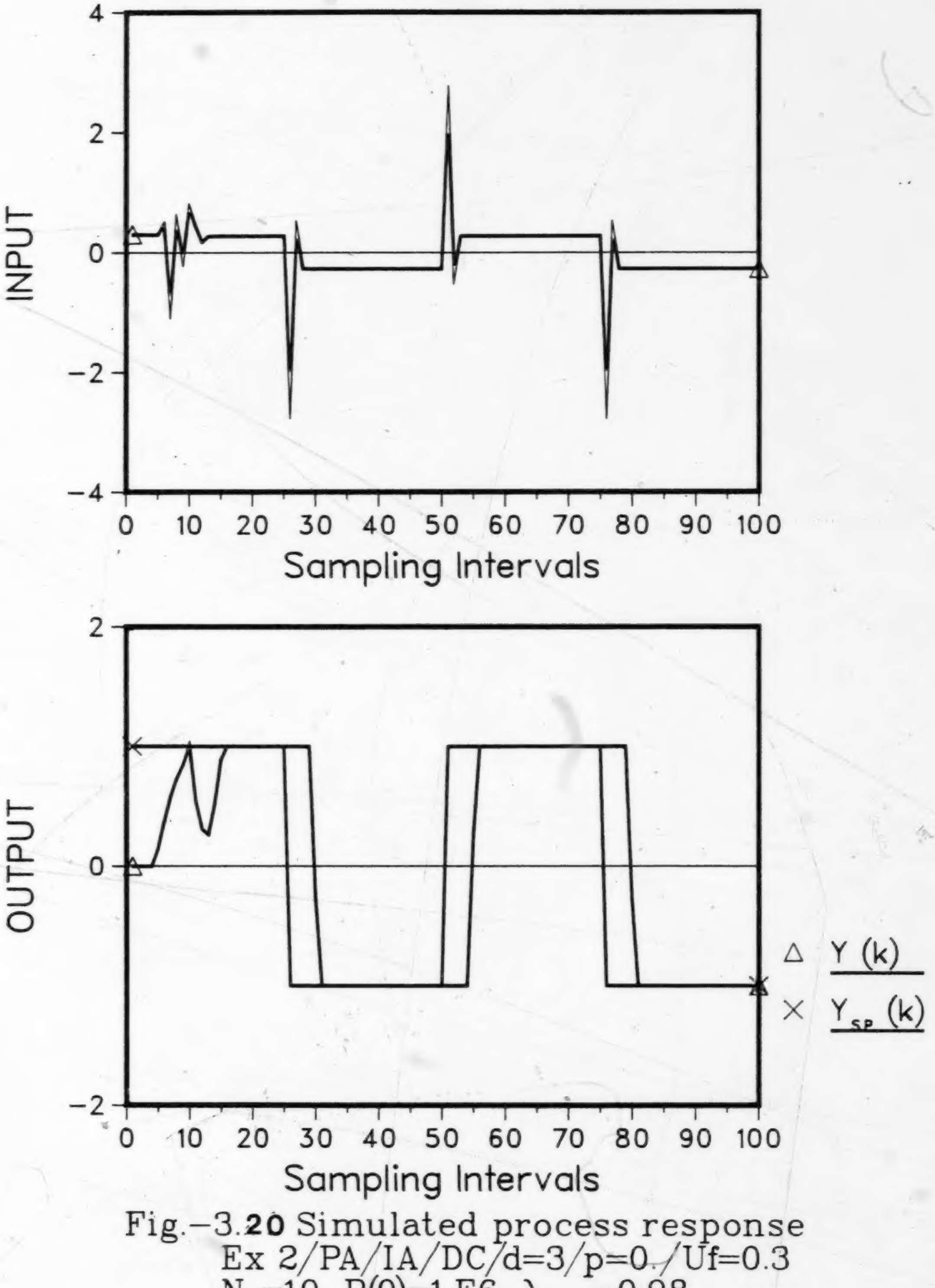


Fig.–3.**20** Simulated process response
Ex 2/PA/IA/DC/d=3/p=0./Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

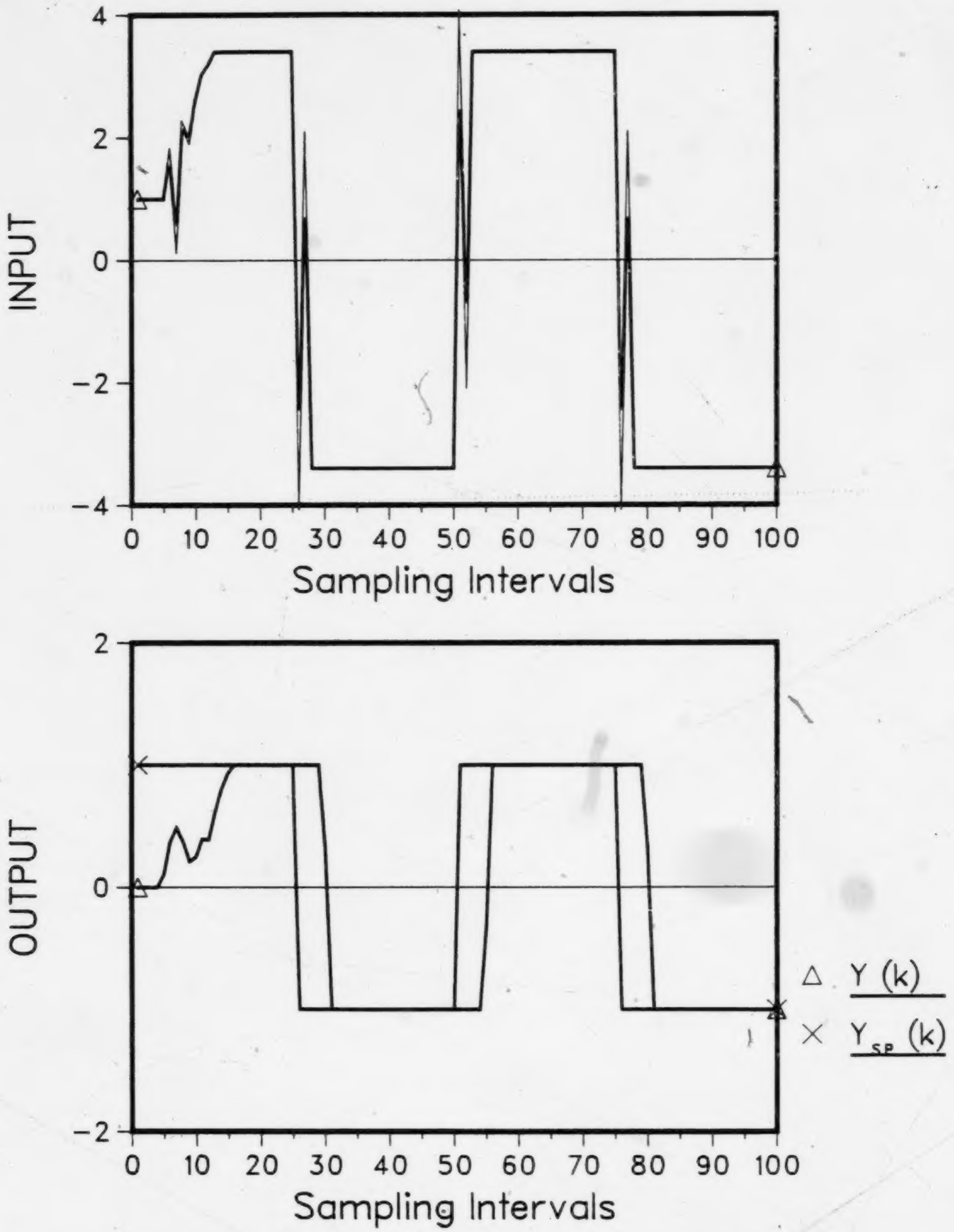


Fig.−3.**21** Simulated process response
Ex 5/PA/IA/DC/d=3/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

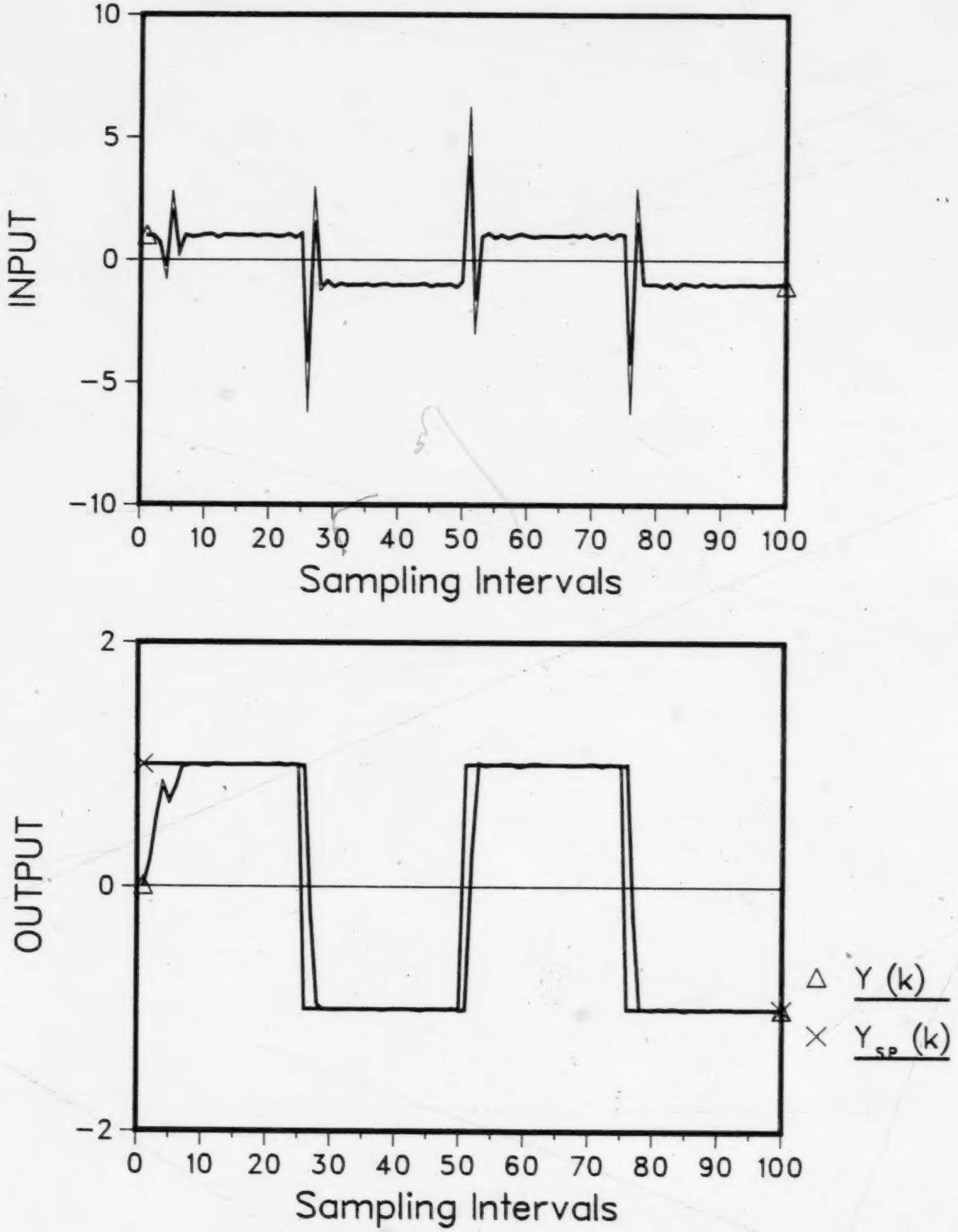


Fig.—3.22 Simulated process response
Ex 1/PA/IA/SC.005/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

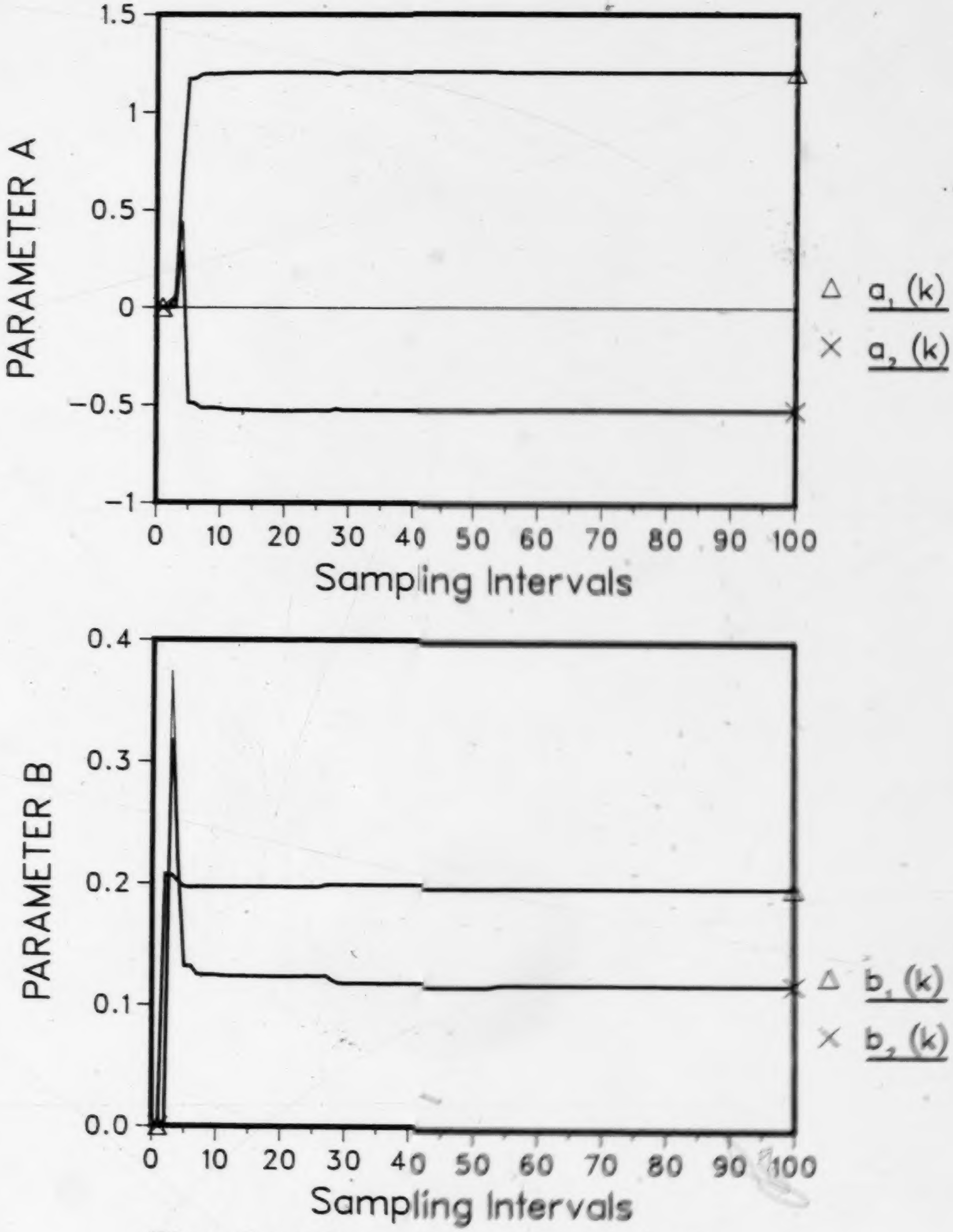


Fig.−3.23 Parameter trajectory
Ex 1/PA/IA/SC.005/d=0/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

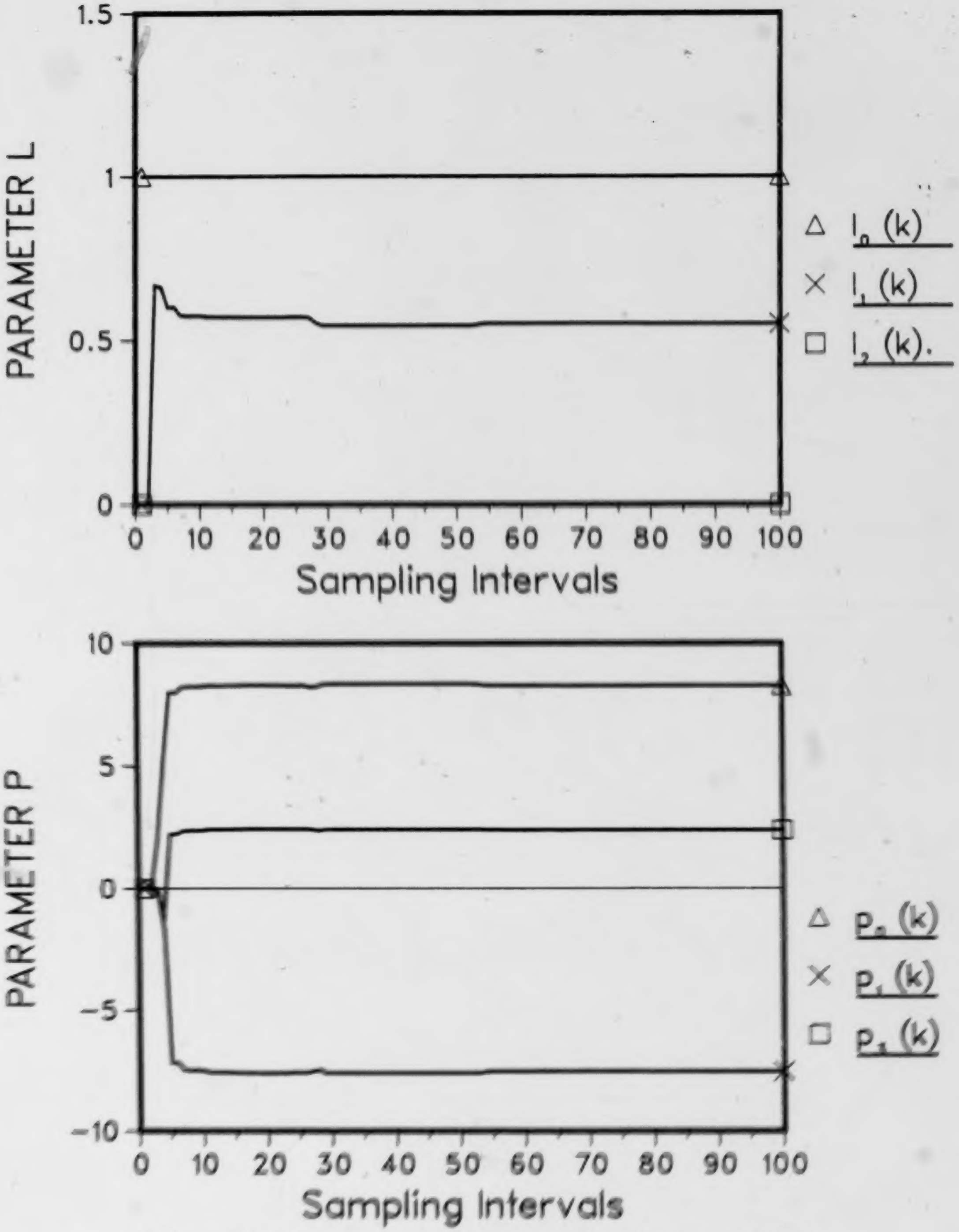


Fig.-3.24 Parameter trajectory of PA
Ex 1/PA/IA/SC.005/d=0/p=0./Uf=1.
N_0=100 P(0)=1.E6 λ_{max}=0.98

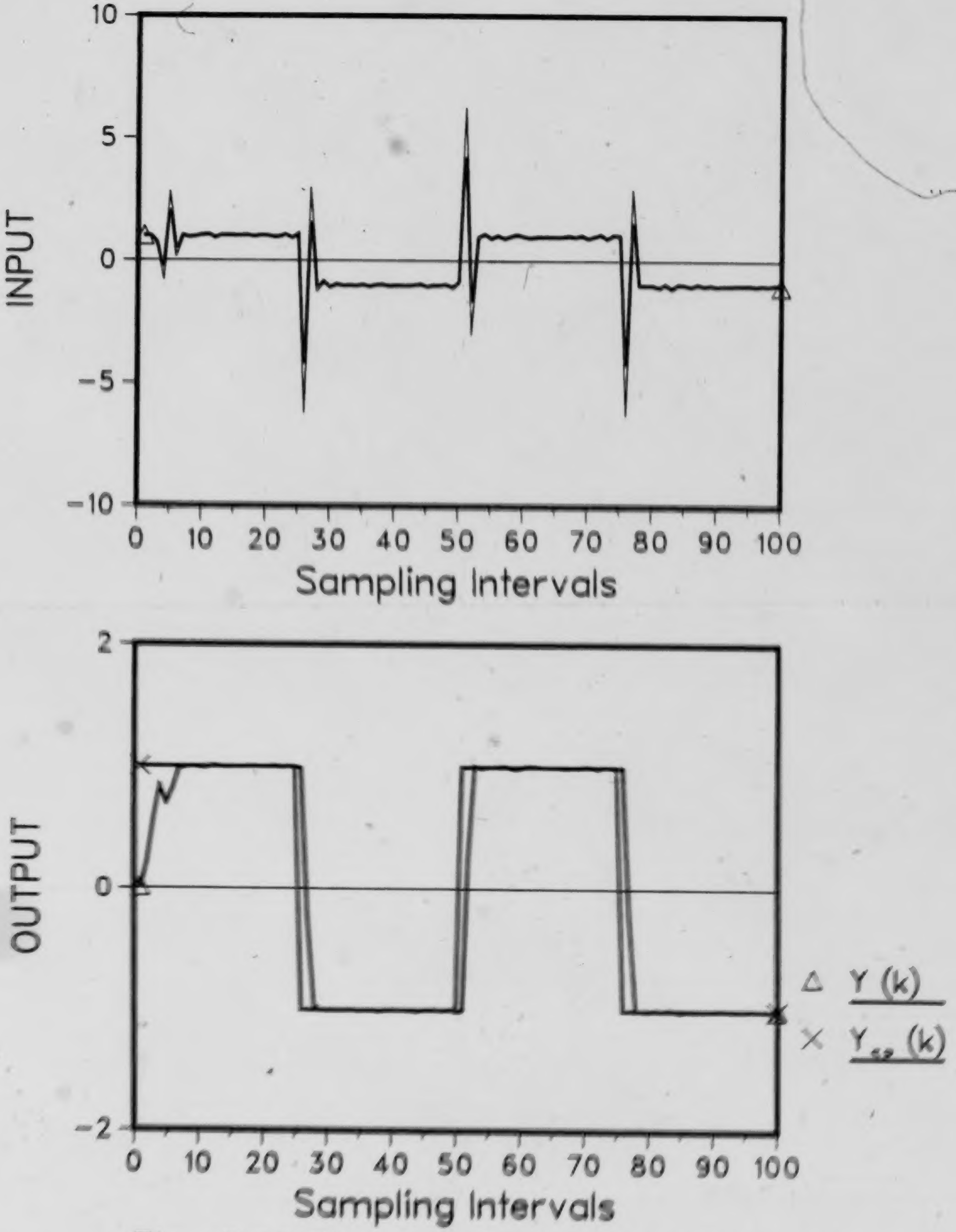
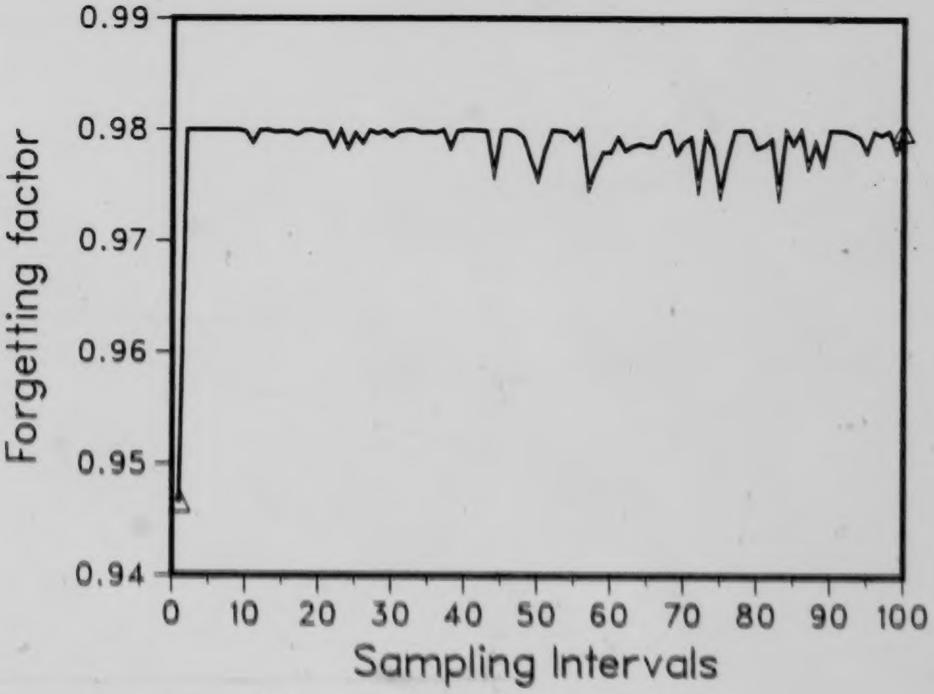


Fig.-3.25 Simulated process response
Ex 1/PA/IA/SC.01/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98



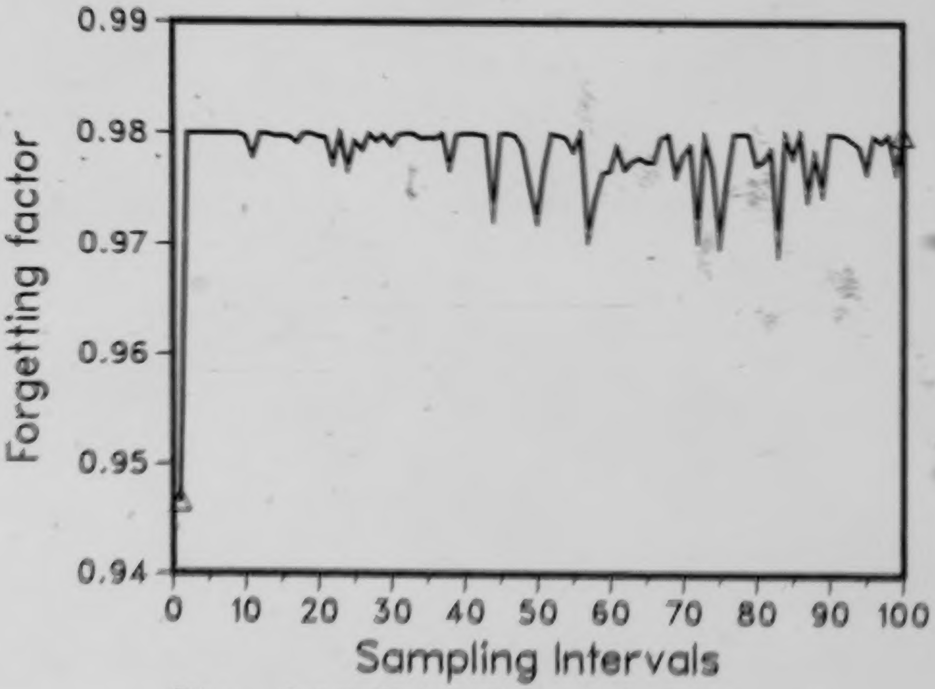


Fig.–3.26 Forgetting factor
Ex 1/PA/IA/SC/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

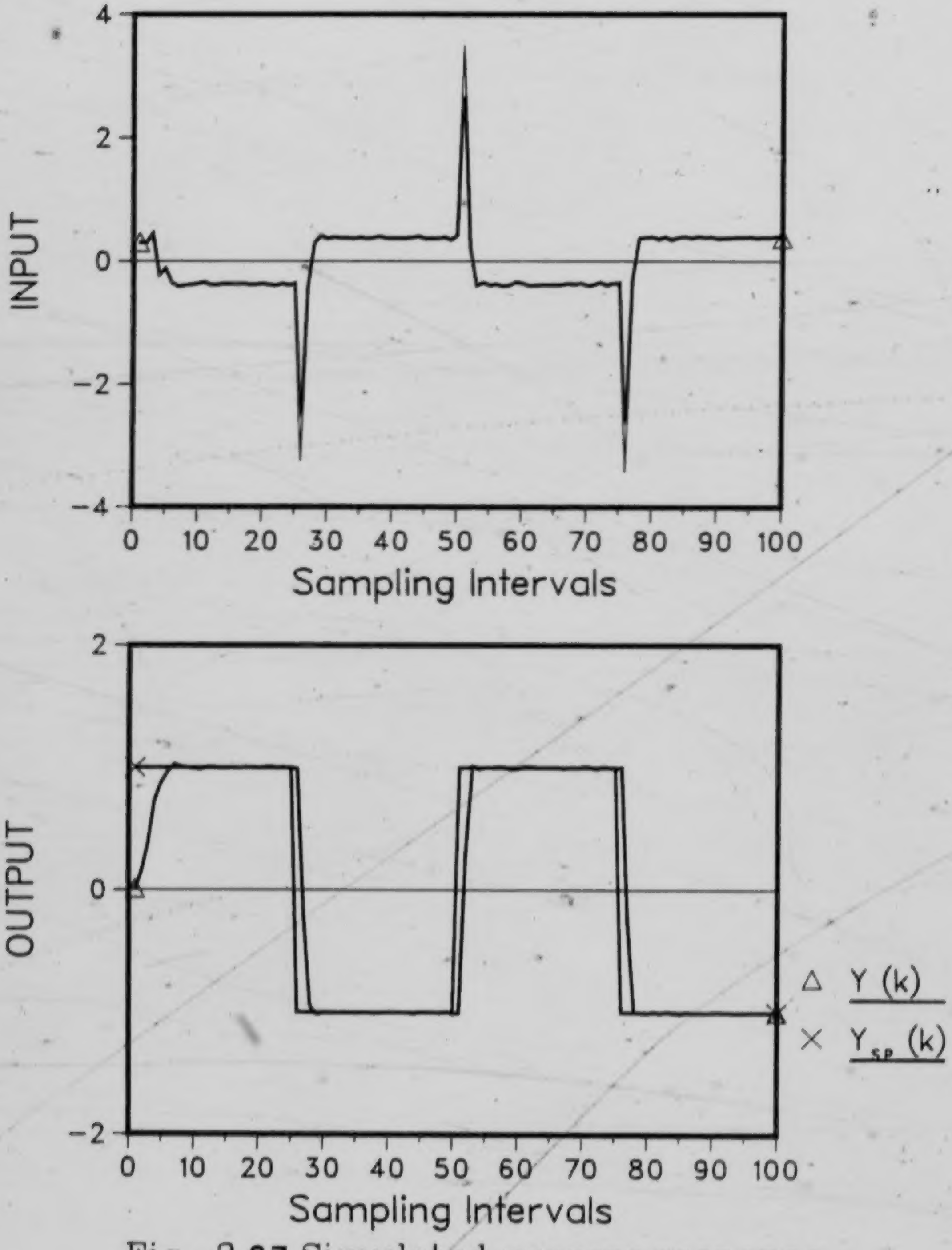


Fig.–3.**27** Simulated process response
Ex 2/PA/IA/SC.005/d=0/p=0./Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

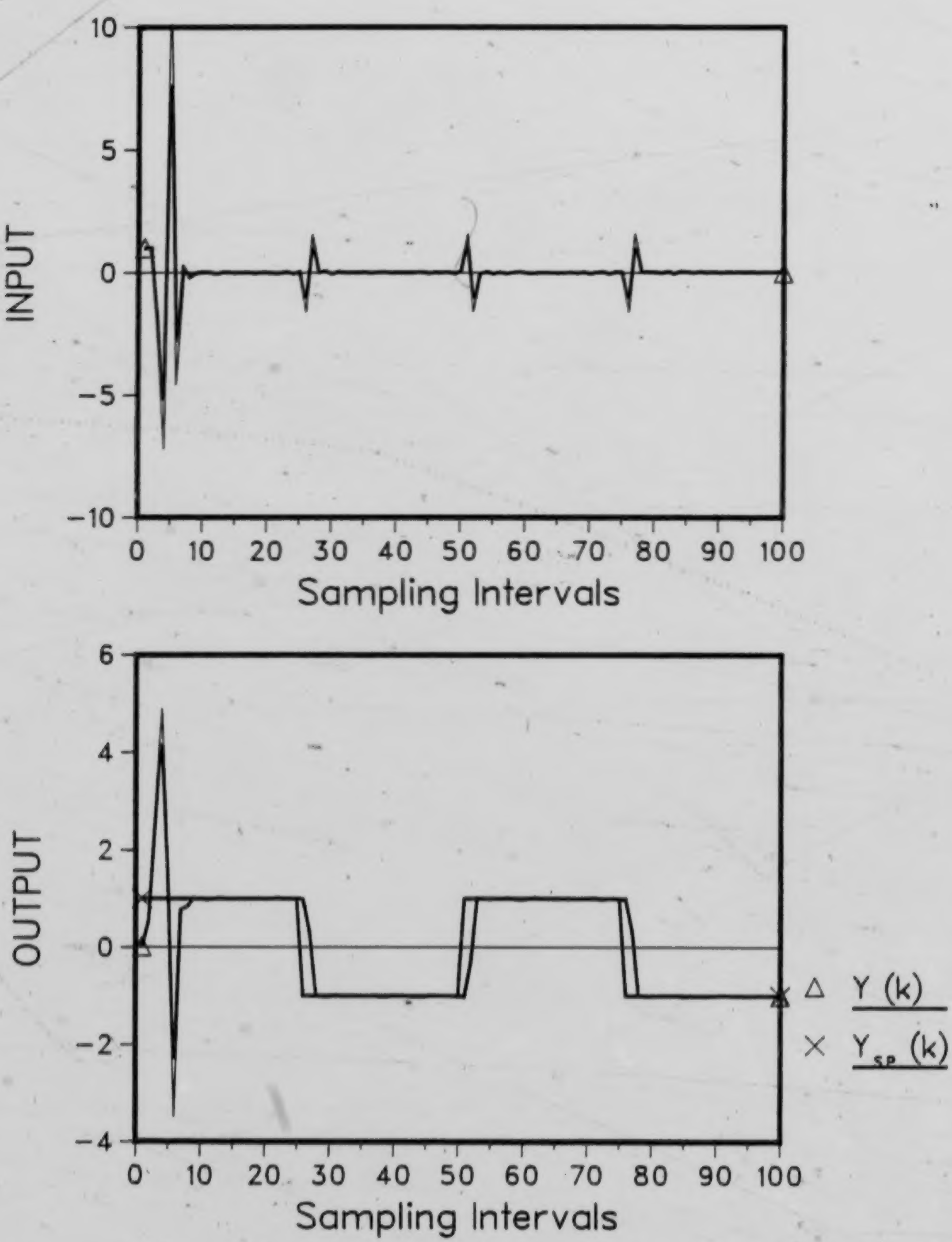


Fig.−3.**28** Simulated process response
Ex 3/PA/IA/SC.005/d=0/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

3.6 Conclusions

In this chapter we analysed a pole-assignment control algorithm which is based on the classical pole positioning techniques.

By using the same mathematical tools as used by Goodwin and Sin (1981), we proved local stability of a closed loop system in the presence of bounded disturbances. The stability analysis is valid for both minimum and nonminimum phase processes. By extending Johnstone's work to the stochastic case, we proved global stability of a closed loop system using the pole-assignment algorithm with disturbances. A modified control law was used to improve the control performance when sharp set point changes occur. The excellent performance of this controller was illustrated by a number of simulation examples in the deterministic, deterministic with time delay and stochastic cases.

4. Pole-assignment with Smith predictor (PASP)

4.1 Introduction

Many control loops contain time delays. Transportation of fluids over long distances, required time for the completion of the sampling and analysis of a measured variable by a measuring device, required time for the development of the actuating signal by the final control element are some of the reasons for the presence of time delays in many processes. The presence of time delays is generally a serious impediment to good process control and operation of a unit. A disturbance entering the system will be detected a significant time after it has entered and upset the system. A control decision will try to regulate a situation that happened some time ago and its effect will be felt by the process some time after the decision has been taken. These are some reasons illustrating the effect of time delays on the process operation.

In the late 1950's, Smith (1957, 1959) developed a compensator for a single delay in a single loop to eliminate the effect of a time delay on the characteristic polynomial of the closed loop system. Moore et. al. (1970) used the analytical solution of the modelling equation of a state space model to predict the value of the state d steps (equivalent to one unit delay) ahead. Alevisakis and Seborg (1973,1974) extended these results to the multivariable case where only one single delay was present. Ogunnaikie and Ray

(1979) have analysed the general case of multivariable systems with multiple delays. They eliminated the time delays in the output variables by predicting certain state variables at various specific times in the future.

In the previous chapter we introduced an explicit adaptive pole-assignment algorithm. Of course, this algorithm can handle time delays but the solution of equation (3.10) at each control interval becomes a real problem when the time delay is large (especially because of the large dimension of the Sylvester matrix). We can imagine the case of controlling the conversion in a polymer reactor by manipulating the flowrate of cold water around its jacket. This process can be easily described by a second order model. Suppose that we use a gas chromatograph as a measuring device (it usually takes twelve minutes to sample and perform the analysis) and that the control interval is half a minute. In this case the time delay will be twenty four sampling intervals for which the dimension of the corresponding Sylvester matrix (equation (3.10)) will be equal to fifty two. It is very obvious that the solution of equation (3.10) at each control interval will restrict the usefulness of this control algorithm.

In next section we shall show how through the use of a simple predictor the adaptive pole-assignment algorithm can become computationally efficient, despite the presence of large time delays. The idea is to use a Smith predictor as it was introduced by Smith (1959).

4.2 Pole-assignment algorithm with dead time compensation

Consider a single input/single output discrete time linear and time invariant process described by

$$A(z^{-1})\, y(k) = z^{-d}\, B(z^{-1})\, u(k) \qquad (4.1)$$

where z^{-1} is the time delay operator, $\{u(k)\}$ and $\{y(k)\}$ are the input and output sequences. The polynomials $A(z^{-1})$ and $B(z^{-1})$ are given by

$$A(z^{-1}) = 1 - a_1\, z^{-1} - \ldots - a_n\, z^{-n} \qquad (4.2)$$

$$B(z^{-1}) = b_{d+1}\, z^{-1} + \ldots + b_{d+m}\, z^{-m} \qquad (4.3)$$

Also, d is the time delay ($d \geq 0$).

Assume that:

A1] d is known

A2] $r = \max\{\, n+1,\ m \,\}$

A3] $A(z^{-1})$ and $B(z^{-1})$ are relatively prime polynomials

In a Smith predictor, the main idea is to compare the reference signal with a measurement signal that carries current and not delayed information. This is shown in the following figure where an integrator, for steady state error elimination, is also used.

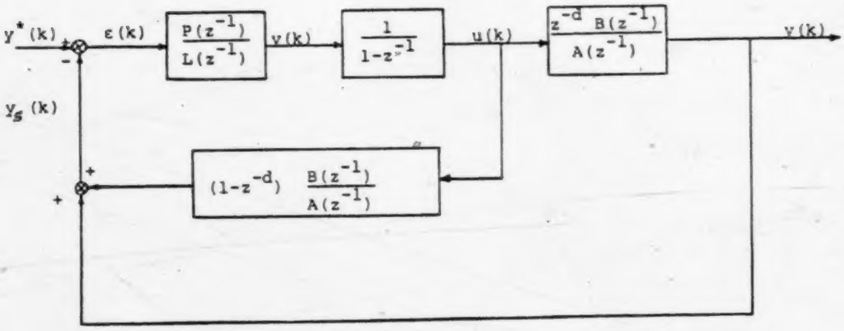

$y^*(k)$
+
-
$\varepsilon(k)$
$\frac{P(z^{-1})}{L(z^{-1})}$
$v(k)$
$\frac{1}{1-z^{-1}}$
$u(k)$
$z^{-d}\frac{B(z^{-1})}{A(z^{-1})}$
$y(k)$
$y_s(k)$
+
+
$(1-z^{-d})\frac{B(z^{-1})}{A(z^{-1})}$

Fig. - 4.1 Closed loop control system with dead time compensation

In the above figure, $y_s(k)$ is the signal which carries current (undelayed) information and is compared with the reference signal. This auxiliary signal, $y_s(k)$, tends to $y(k)$ after parameter convergence has been established.
From figure 4.1, we can easily find that the transfer function between $y(k)$ and $y^*(k)$ is given by

$$\frac{y(k)}{y^*(k)} = \frac{z^{-d}\,P(z^{-1})\,B(z^{-1})}{(1-z^{-1})\,A(z^{-1})\,L(z^{-1}) + B(z^{-1})\,P(z^{-1})} \qquad (4.4)$$

Notice that the time delay term z^{-d} has been eliminated in the characteristic polynomial.

If we want to assign the poles at prespecified locations, described by the stable monic polynomial $A^*(z^{-1})$ of degree 2r-1, then we have to solve the following equation

$$(1-z^{-1})\,A(z^{-1})\,L(z^{-1}) + B(z^{-1})\,P(z^{-1}) = A^*(z^{-1}) \tag{4.5}$$

with respect to the polynomials $L(z^{-1})$ and $P(z^{-1})$. The polynomials $L(z^{-1})$ and $P(z^{-1})$ are again of order r-1. The control law is given by

$$L(z^{-1})\,v(k) = P(z^{-1})\,[y^*(k) - y_s(k)] \tag{4.6}$$

$$(1-z^{-1})\,u(k) = v(k) \tag{4.7}$$

In a closed loop contol system the process parameters are estimated recursively on line and are used instead of the true ones. In such a case, the following equations hold

Process: $$y(k) = \underline{\psi}(k-1)^t\,\underline{\theta}_0 \tag{4.8}$$

Model: $$\hat{y}(k) = \underline{\psi}(k-1)^t\,\underline{\theta}(k-1) \tag{4.9}$$

Auxiliary output: $$y_s(k) = y(k) + (1-z^{-d})\,\frac{\hat{B}(z^{-1},k)}{\hat{A}(z^{-1},k)}\,u(k) \tag{4.10}$$

Prediction error: $$e(k) = y(k) - \hat{y}(k) \tag{4.11}$$

Pole-assignment equation: $$(1-z^{-1})\,\hat{A}(z^{-1},k)\,\hat{L}(z^{-1},k) + \\ + \hat{B}(z^{-1},k)\,\hat{P}(z^{-1},k) = A^*(z^{-1}) \tag{4.12}$$

Control law: $$\hat{L}(z^{-1},k)\,v(k) = \hat{P}(z^{-1},k)\,. \\ .\,[y^*(k) - y_s(k)] \tag{4.13}$$

and
$$(1-z^{-1})\,u(k) = v(k) \tag{4.14}$$

where

$$\underline{\psi}(k-1)^t = [y(k-1)\ \dots\ y(k-r)\ u(k-d-1)\ \dots\ u(k-d-r)]$$

$$\underline{\theta}_o(k-1)^t = [a_1\ \dots\ a_t\ b_{d+1}\ \dots\ b_{d+r}]$$

$$a_i = 0 \text{ if } i>n$$

$$b_i = 0 \text{ if } i>m$$

$$\underline{\theta}(k)^t = [\hat{a}_1(k)\ \dots\ \hat{a}_r(k)\ \hat{b}_{d+1}(k)\ \dots\ \hat{b}_{d+r}(k)]$$

All the other quantities are as they have been defined in the previous chapter. It can be easily seen that the order of the Sylvester matrix in equation (4.12) will not depend on the magnitude of the time delay. The use of the Smith predictor eliminates the time delay from the characteristic equation of the closed loop transfer function and makes the solution of the pole-assignment algorithm computationally more efficient. The performance of the compensator depends on our knowledge about the process and the time delay. Perfect compensation will be obtained only when there will be no modelling mismatch and the time delay is exactly known. The larger the modelling error, the less effective is the compensator. The error in estimating the time delay is more crucial than the modelling error for effective dead time compensation.

The stability of the closed loop system with dead time compensation will be considered in the next section.

Stability analysis

By using the same ideas, as in section 3.3, we shall initially prove local stability of the closed loop control system with dead time compensation. For the sake of simplicity we shall consider the deterministic case. The stability result is stated in the following theorem.

Theorem 4.1

Assume that:

A1] d is known

A2] $r = \max\{n+1, m\}$ is known

A3] $y^*(k)$ is a bounded reference signal

A4] $A^*(z^{-1})$ is a stable monic polynomial of degree $2r-1$

A5] Lemma 3.2 is satisfied

A6] The identification scheme stated by equations (2.24) to (2.30) is used

Then, the control algorithm (4.8) to (4.14) is stable in the sense that

a. $\{u(k)\}$ is bounded

b. $\{y(k)\}$ is bounded

Proof

We define

$$\hat{A}\,\hat{B} = \sum_i \sum_j \hat{a}_i(k)\,\hat{b}_j(k)\,z^{-i-j} = \hat{B}\,\hat{A} \qquad (4.15)$$

$$\hat{A}.\hat{B} = \sum_i \sum_j \hat{a}(k)\,\hat{b}_j(k-1)\,z^{-i-j} \neq \hat{B}.\hat{A} \qquad (4.16)$$

$$\hat{\hat{B}} = \hat{B}(z^{-1},k-1) \qquad (4.17)$$

We also have

$$A\ y(k) = z^{-d}\ B\ u(k) \qquad (4.18)$$

$$\hat{L}\ v(k) = \hat{P}\ y^*(k) - \hat{P}\ y_s(k) \qquad (4.19)$$

$$(1-z^{-1})\ u(k) = v(k) \qquad (4.20)$$

$$(1-z^{-1})\ \hat{A}\ \hat{L} + \hat{B}\ \hat{P} = A^* \qquad (4.21)$$

$$\begin{aligned}
e(k) &= y(k) - \hat{y}(k) \\
&= y(k) - \underline{\psi}(k-1)^t\ \underline{\theta}(k-1) \\
&= y(k) - \hat{a}_1(k-1)\ y(k-1) - \ldots - \hat{a}_r(k-1)\ y(k-r) - \\
&\quad - \hat{b}_{d+1}(k-1)\ u(k-d-1) - \ldots - \hat{b}_{d+r}\ u(k-d-r)
\end{aligned}$$

or

$$e(k) = \hat{A}\ y(k) - z^{-d}\ \hat{\hat{B}}\ u(k) \qquad (4.22)$$

We define the auxiliary signal w(k) as

$$w(k) = \hat{A}.\hat{P}\ y^*(k) \qquad (4.22)$$

Equation (4.23) gives

$$\begin{aligned}
w(k) &= \hat{A}.\hat{P}\ y^*(k) \\
&= \hat{A}.\hat{L}\ v(k) + \hat{A}.\hat{P}\ y_s(k) \\
&= \hat{A}\ \hat{L}\ v(k) + [\hat{A}.\hat{L} - \hat{A}\ \hat{L}]\ v(k) \\
&\quad + \hat{A}\ \hat{P}\ y_s(k) + [\hat{A}.\hat{P} - \hat{A}\ \hat{P}]\ y_s(k) \\
&= \hat{A}\ \hat{L}\ v(k) + \hat{B}\ \hat{P}\ u(k) + \hat{P}\ e(k) \\
&\quad + [\hat{A}.\hat{L} - \hat{A}\ \hat{L}]\ v(k) + [\hat{A}.\hat{P} - \hat{A}\ \hat{P}]\ y_s(k) \\
&\quad - \hat{B}\ \hat{P}\ u(k) - \hat{P}\ e(k) + \hat{A}\ \hat{P}\ y_s(k) \\
&= (1-z^{-1})\ \hat{A}\ \hat{L}\ u(k) + \hat{B}\ \hat{P}\ u(k) + \hat{P}\ e(k) \\
&\quad + [\hat{A}.\hat{L} - \hat{A}\ \hat{L}]\ (1-z^{-1})\ u(k) + [\hat{A}.\hat{P} - \hat{A}\ \hat{P}]\ y(k) \\
&\quad + [\hat{A}.\hat{P} - \hat{A}\ \hat{P}]\ (1-z^{-d})\ \hat{B}/\hat{A}\ u(k) - \hat{B}\ \hat{P}\ u(k) \\
&\quad - \hat{P}.[A\ y(k) - z^{-d}\ \hat{\hat{B}}\ u(k)] + \hat{A}\ \hat{P}\ y_s(k)
\end{aligned}$$

or

$$\begin{aligned}
w(k) &= A^*\ u(k) + \hat{P}\ e(k) \\
&\quad + [\hat{A}.\hat{L} - \hat{A}\ \hat{L}]\ (1-z^{-1})\ u(k)
\end{aligned}$$

$$+ [\hat{A}.\hat{P} - \hat{A}\ \hat{P}]\ y(k)$$

$$+ [\hat{A}.\hat{P} - \hat{A}\ \hat{P}]\ (1-z^{-d})\ \hat{B}/\hat{A}\ u(k)$$

$$+ [z^{-d}\ \hat{P}\ \hat{\hat{B}} - \hat{P}\ \hat{B}]\ u(k)$$

$$- [\hat{P}\ \hat{\hat{A}}\ y(k) - \hat{P}\ \hat{A}\ y_s(k)]$$

After convergence of the parameters has been established, the terms in the square brackets become zero and the above equation reduces to the following

$$A^*\ u(k) = w(k) - \hat{P}\ e(k) \tag{4.24}$$

By defining the auxiliary signal x(k) as

$$x(k) = \hat{B}.\hat{P}\ y^*(k)$$

we can prove that

$$A^*\ y(k) = x(k) + \hat{P}\ e(k) \tag{4.25}$$

after convergence of the parameters has been established. Now, we can use the same arguments used for the proof of Theorem 3.3 to show that the regressor is linearly bounded by the prediction error. Having proven this, we can use equation (2.60) and the key technical lemmma of Goodwin and Sin (1981) to show that {u(k)} and {y(k)} are bounded sequences which proves our theorem. The stochastic case can be treated as in section 3.4.

Since Theorem 4.1 is based on Lemma 3.2 the stability results stated in it are local in nature. By using, initially, a fixed controller which stabilizes the closed loop system and a persistently exciting reference signal, we can change the local stability results to global ones. The mathematical treatment of the problem is quite straightforward and follows the same lines as in sections

3.4 and 3.5.

4.3 PID controller based on pole-assignment

The PID controller is the most common controller in the industry even though in some cases a great deal of time and effort is required to tune its parameters. On the other hand there are a number of adaptive controllers which have been shown to perform better than the fixed gain PID controller in many cases. Despite this, only few of them have been applied to an industrial environment. The main reason for this is the complicated structure and unfamiliarity with these adaptive schemes. Because of this, there is a lot of incentive in designing adaptive PID controllers that are able to adjust their gains according to the process requirements.

In the past, some authors (Wittenmark (1979), Isermann (1981), Andreiev (1981), Banyasz and Keviczky (1982), Cameron and Seborg (1982), Gawthrop (1980, 1982)) have proposed adaptive PID controllers. In most of the cases, the adaptive PID controllers have been derived for processes without time delays.

In this section we shall derive, based on certain conditions, an adaptive PID controller based on the pole-assignment control algorithm with the dead time compensation feature discussed earlier in section 4.2 .
For the following analysis, we require that

A1] $n = 2$

A2] $m = 2$

A3] d is known

In this case

$$r=3$$

while

$$L(z^{-1}) = 1 + l_1 z^{-1} + l_2 z^{-2}$$

$$P(z^{-1}) = p_0 + p_1 z^{-1} + p_2 z^{-2}$$

$$A^*(k) = 1 + a^*_1 z^{-1} + \dots + a^*_5 z^{-5}$$

Also, the control law will be given by

$$(1 + l_1 z^{-1} + l_2 z^{-2})\, v(k) = (p_0 + p_1 z^{-1} + p_2 z^{-2}) \cdot [y^*(k) - y_s(k)]$$

and

$$(1-z^{-1})\, u(k) = v(k)$$

The above two equations can be combined to give

$$(1 + l_1 z^{-1} + l_2 z^{-2}) \nabla u(k) = (p_0 + p_1 z^{-1} + p_2 z^{-2}) \cdot \epsilon(k) \qquad (4.26)$$

where

$$\nabla u(k) = u(k) - u(k-1) \qquad (4.27)$$

$$e(k) = y^*(k) - y_s(k)$$

At this point, we make the following assumption that

$$1 + l_1 z^{-1} + l_2 z^{-2} = 1 + l_1 + l_2 = \Sigma l_i \qquad (4.28)$$

The above assumption is valid for steady state conditions or for the case when the input changes very little between successive intervals. Under this assumption, equation (4.26) gives

$$\nabla u(k) = (\gamma_0 + \gamma_1 z^{-1} + \gamma_2 z^{-2}) \, \epsilon(k) \tag{4.29}$$

where

$$\gamma_0 = \frac{p_0}{\Sigma l_i}$$

$$\gamma_1 = \frac{p_1}{\Sigma l_i}$$

$$\gamma_2 = \frac{p_2}{\Sigma l_i}$$

We can easily see that equation (4.29) is the equation of a velocity type PID controller. The velocity-type PID controller is given by

$$\begin{aligned}\nabla u(k) = k_m \, [&\epsilon(k) - \epsilon(k-1) + \tau_s/\tau_i \, \epsilon(k) \\ &+ \tau_d/\tau_s \, (\epsilon(k) - 2 \, \epsilon(k-1) + \epsilon(k-2))]\end{aligned} \tag{4.30}$$

or in a more compact form

$$\begin{aligned}\nabla u(k) = k_m \, [&(1 + \tau_d/\tau_s + \tau_s/\tau_i) \\ &- (1 + 2 \, \tau_d/\tau_s) \, z^{-1} \\ &+ \tau_d/\tau_s \, z^{-2}] \, \epsilon(k)\end{aligned} \tag{4.31}$$

where k_m is the proportional gain, τ_i is the integral gain, τ_d is the derivative gain and τ_s is the sampling time.

By direct comparison between equations (4.29) and (4.31) we get

$$k_m \, (1 + \tau_d/\tau_s + \tau_s/\tau_i) = \gamma_0$$

$$k_m \, (1 + 2 \, \tau_d/\tau_s) = - \gamma_1$$

$$k_m \, \tau_d/\tau_s = \gamma_2$$

The above system of equations can be solved with respect to k_m, τ_d/τ_s and τ_s/τ_i to give

$$k_m = -\ (\gamma_1 + 2\ \gamma_2) \tag{4.32}$$

$$\hat{\tau}_d = \frac{\tau_d}{\tau_s} = -\ \frac{\gamma_0 + \gamma_1 + \gamma_2}{\gamma_1 + 2\ \gamma_2} \tag{4.33}$$

$$\hat{\tau}_i = \frac{\tau_s}{\tau_i} = -\ \frac{\gamma_2}{\gamma_1 + 2\ \gamma_2} \tag{4.34}$$

where $\hat{\tau}_d$ and $\hat{\tau}_i$ can be considered as reduced derivative and integral gains respectively.

If we want to avoid the derivative kick when a set point change occurs, the following control equation can be used

$$\begin{aligned} \nabla u(k) = k_m\ [&y_s(k-1) - y_s(k) + \hat{\tau}_i(k)\ \epsilon(k)\ + \\ &+ \hat{\tau}_d\ (2\ y_s(k-1) - y_s(k) - y_s(k-2))] \end{aligned} \tag{4.35}$$

In brief, we can say that a PID controller has been derived based on pole-assignment techniques. The controller has been derived for a particular class of processes and it can handle large but known time delays. From equations (4.32) to (4.35) it can be seen that the calculation of the control action does not require knowledge of the sampling time. This does not mean that the sampling time does not affect the control action calculated from this control law. There is an indirect effect of the sampling time on the control action u(k) through the l's and p's parameters.

4.4 Simulation results

In this section, three models from those mentioned in the previous chapter will be considered to illustrate the performance of the pole-assignment with the dead-time compensation feature algorithm under different cases. The same models will also be used to illustrate the performance of the adaptive PID controller mentioned in section 4.4. These models are:

$$\text{Model 1:} \quad G(z^{-1}) = \frac{z^{-d}\,(0.4809\, z^{-1} + 0.2725\ ^{-2})}{1 - 1.0382\, z^{-1} + 0.2466\, z^{-2}}$$

$$\text{Model 2:} \quad G(z^{-1}) = \frac{z^{-d}\,(0.2\, z^{-1} + 0.12\, z^{-2})}{1 - 1.2\, z^{-1} + 0.52\, z^{-2}}$$

$$\text{Model 3:} \quad G(z^{-1}) = \frac{z^{-d}\,(0.1\, z^{-1} + 0.2\ ^{-2})}{1 - 0.7\, z^{-1} + 0.72\, z^{-2}}$$

In all the simulation runs, we have considered a closed loop desired polynomial $A^*(z^{-1})$ of the form:

$$A^*(z^{-1}) = 1 + a_1^*\, z^{-1}$$

By assuming a first order closed loop desired polynomial we can manipulate the location of only one pole. The rest of the poles are cancelled with the closed loop zeros. The initial parameter vector was chosen to be equal to zero while a fixed gain controller was initially used.

It can be seen, from figure 4.2, that the control law (4.13) to (4.14) creates large overshoots when a set point change occurs. By modifying the control law, as in the previous chapter, we can greatly improve the performance of the controller (figure 4.3). The fast convergence of the process and control parameters to their true values can be seen in figures 4.4 and 4.5 . Process input and output trajectories for models 2 and 3 are shown in figures 4.7 and 4.8 . The corresponding parameter trajectories show fast convergence, as in figures 4.4 and 4.5, and is not reproduced here for the sake of brevity. Figure 4.6 has been drawn for the first simulation example when the time delay is three sampling intervals. This figure illustrates the fact that the performane of the pole-assignment algorithm with the dead time compensation feature compares very well with that of the simple pole-assignment algorithm (figure 3.9) discussed in the previous chapter. By using a Smith predictor, the big advantage is that the order of the controller does not depend on the order of the time delay. This makes the control algorithm computationally more efficient. The same simulation examples have been used to illustrate the performance of the controller in a stochastic environment. The noise considered was a Gaussian noise with zero mean and variance of 0.01 . Figures 4.9 to 4.13 show the performance of the controller and the process and controller parameter trajectories for the simulation examples. It can be said that the presence of noise deteriorates the control performance only during the initial transient state. The presence of noise deteriorates and delays the convergence of the parameters. This makes the Smith predictor less effective and consequently affects the whole control algorithm. The control performance is improved as the convergence of the process parameters improves.

Figures 4.13 to 4.26 illustrate the performance of the PID controller. We have considered the deterministic case without time delay, the deterministic case with a time delay of five sampling intervals and the stochastic time delay free case. The noise considered was gaussian with zero mean and variance of 0.01 . Figures 4.14 to 4.18 are referred to the deterministic time delay free case. The performance of the PID controller can be considered as excellent for all the simulation examples (minimum or nonminimum phase systems). Since the derivation of the PID controller is based on assumption (4.28), its performance deteriorates when sudden set point changes occur but still remains satisfactory. It must be mentioned that the performance of this controller heavily depends on the location of the pole. This is true since the pole location affects the response of the system (fast or slow) and this may violate the assumption (4.28). Figures 4.19 to 4.21 illustrate the performance of the adaptive PID controller for the deterministic with a time delay of five sampling intervals case. The presence of the delay affects only the process output. The auxiliary output $y_1(k)$ is exactly the same as the process output in the time delay free case after parameter convergence has been established. Since the time delay has been removed from the closed loop transfer function, the gains of the controller are identical to the gains of the system without the time delay. Figures 4.22 to 4.26 illustrate the performance of the adaptive PID controller in the stochastic case. In this case the time delay was zero. The performance of the controller for all the simulation examples can be considered as excellent.

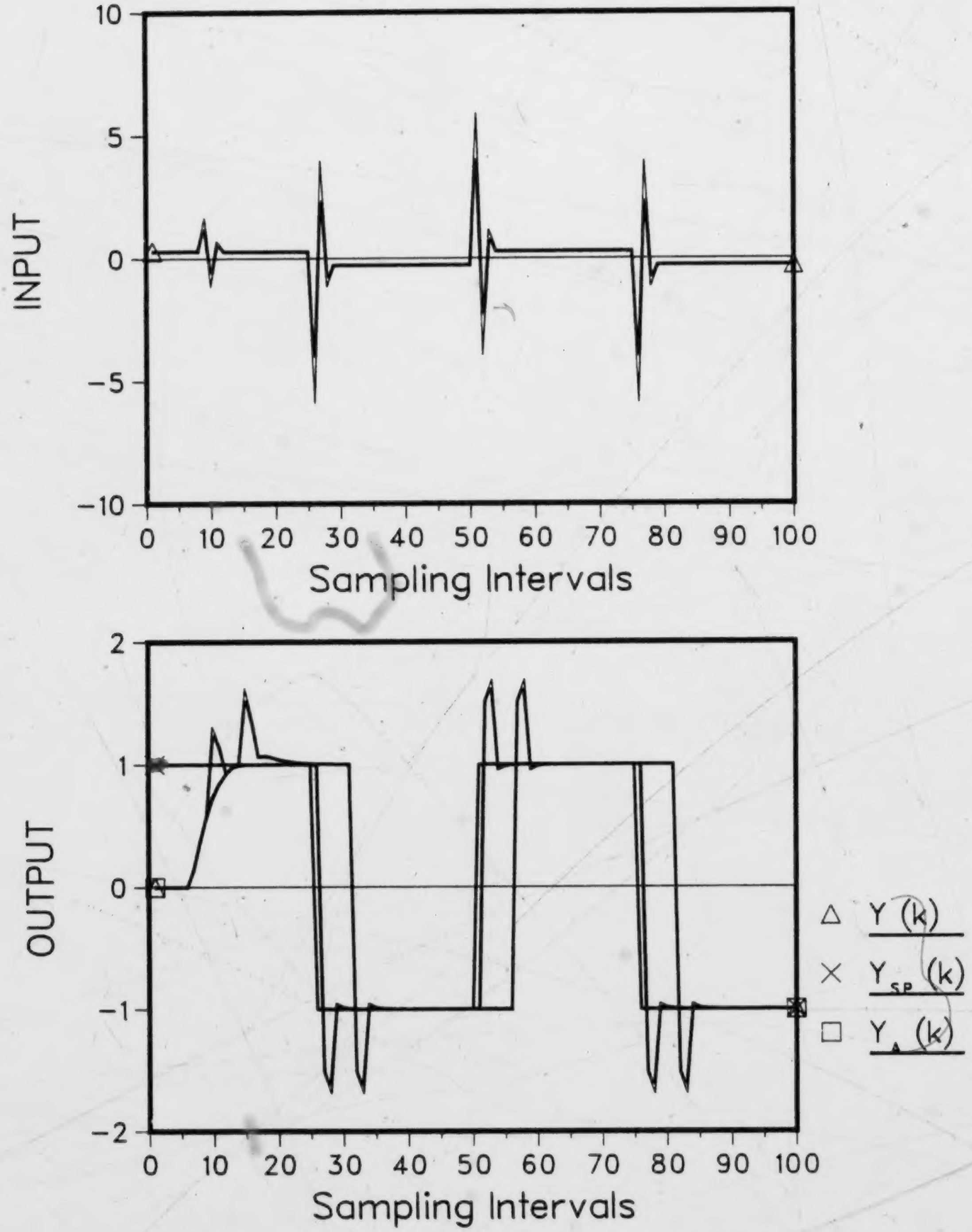


Fig.−4.2 Simulated process response
Ex 1/PASP/IA/DC/d=5/p=0./Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

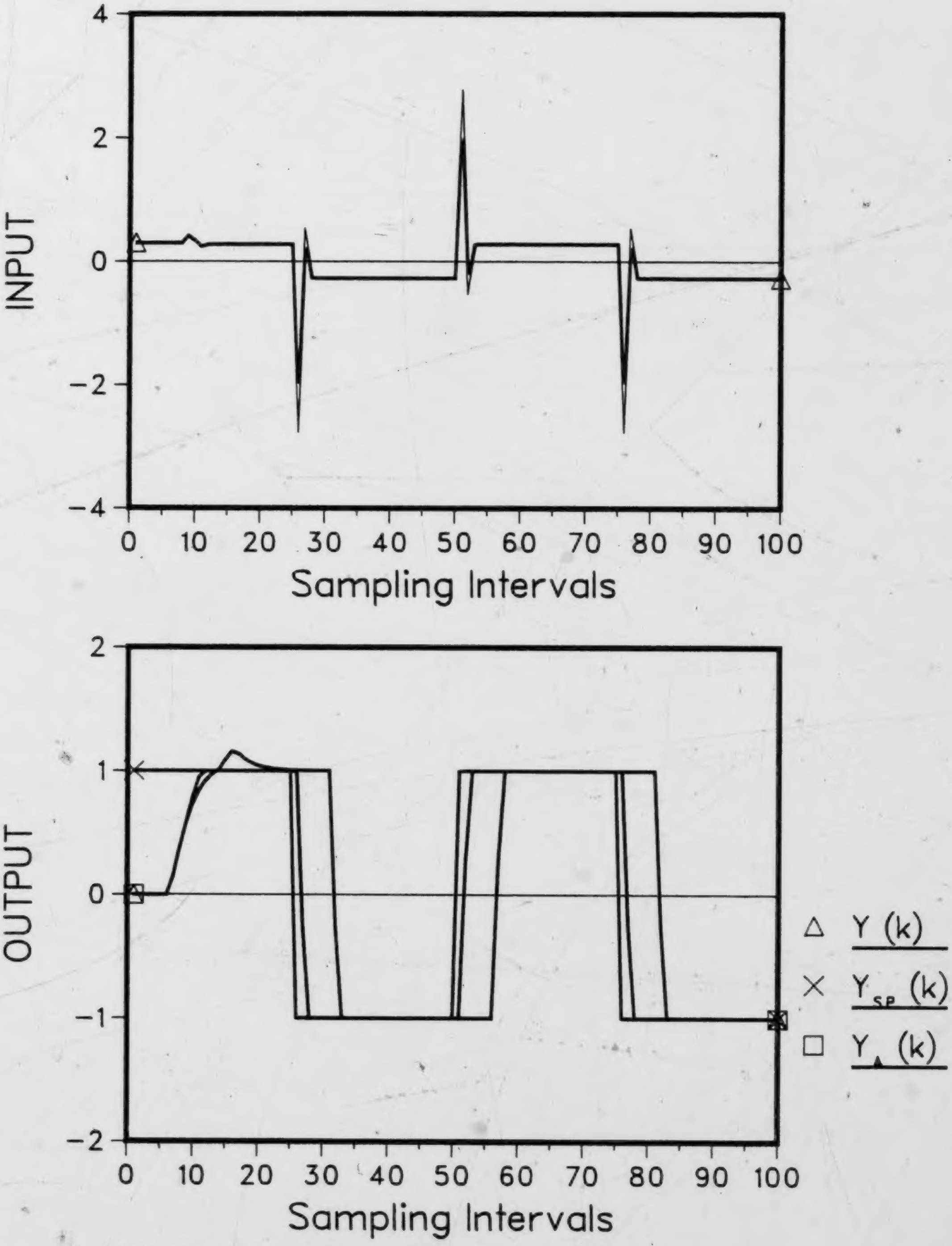


Fig.−4.3 Simulated process response
Ex 1/PASP/IA/DC/d=5/p=0./Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

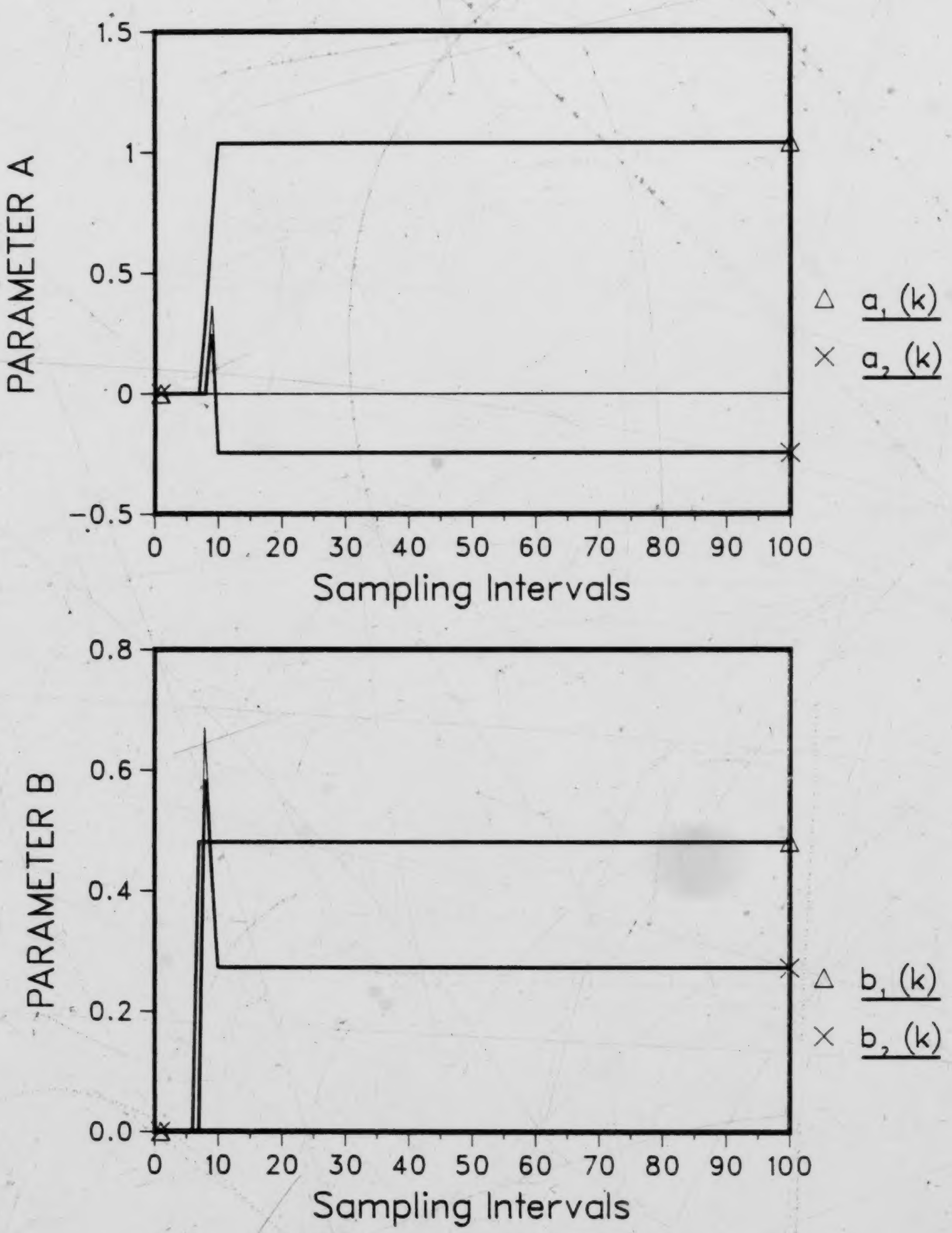


Fig.−4.4 Parameter trajectory
Ex 1/PASP/IA/DC/d=5/p=0./Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

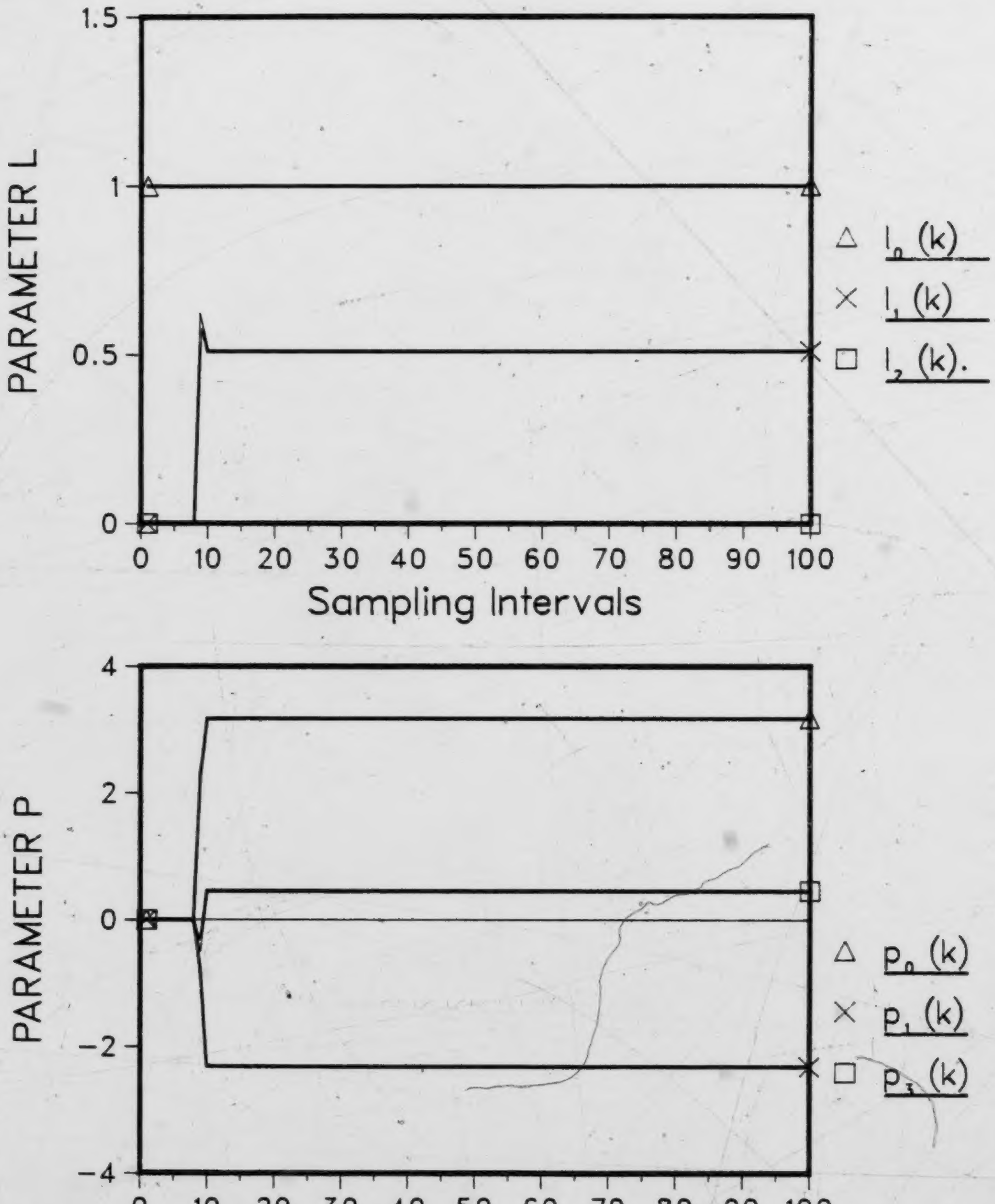


Fig.—4.5 Parameter trajectory of PA
Ex 1/PASP/IA/DC/d=5/p=0./Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

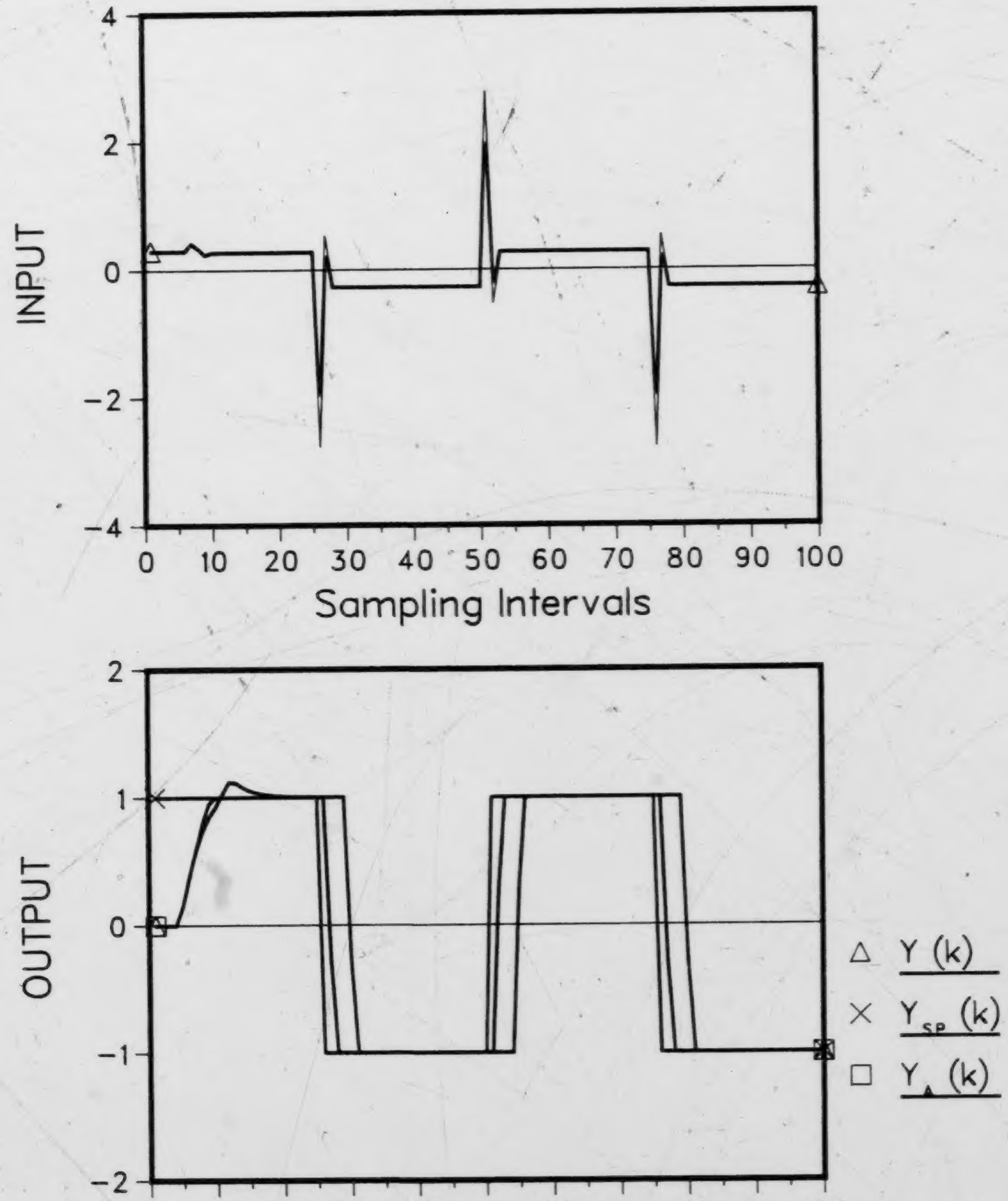


Fig.—4.6 Simulated process response
Ex 1/PASP/IA/DC/d=3/p=0./Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

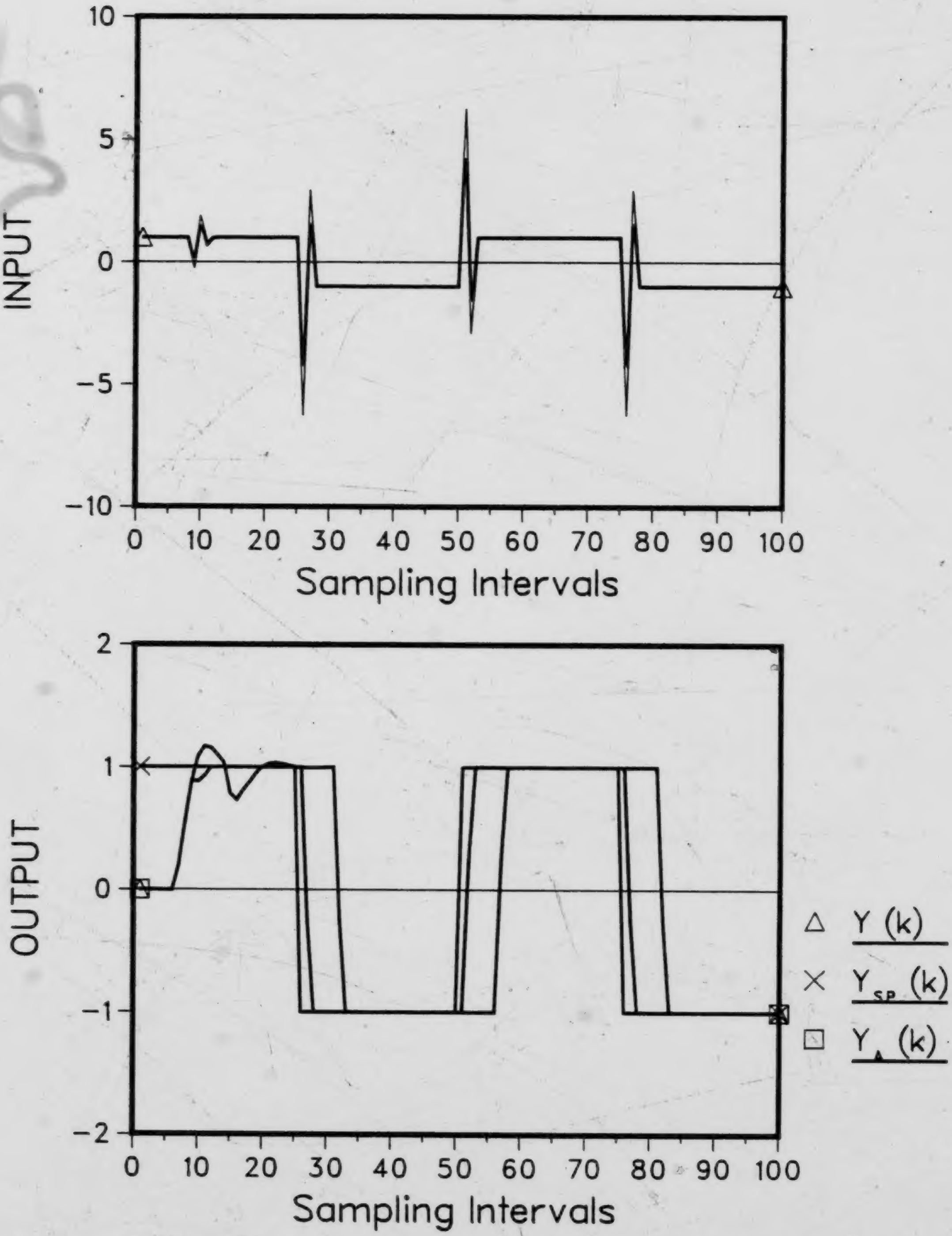


Fig.–4.7 Simulated process response
Ex 2/PASP/IA/DC/d=5/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

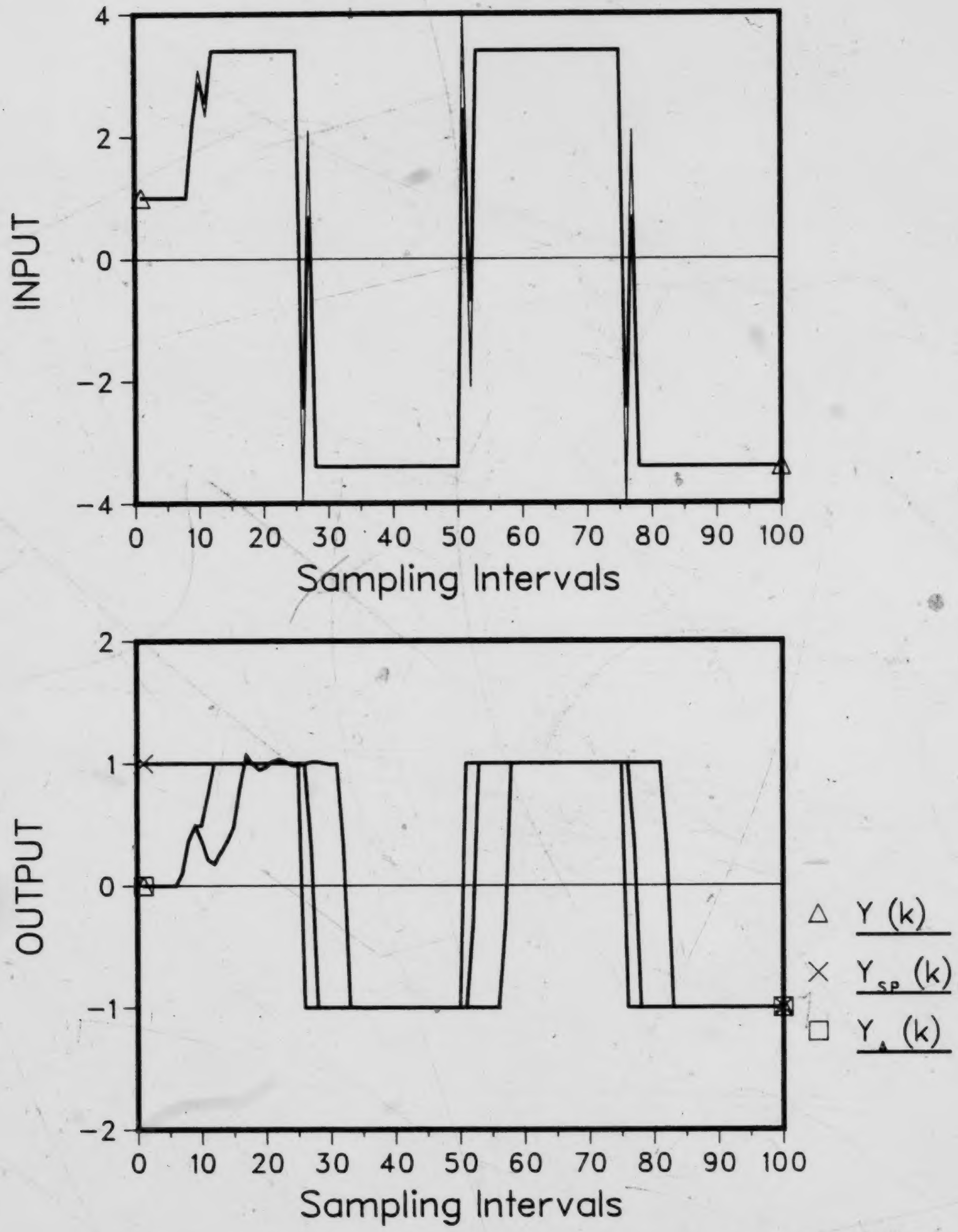


Fig.—4.8 Simulated process response
Ex 3/PASP/IA/DC/d=5/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

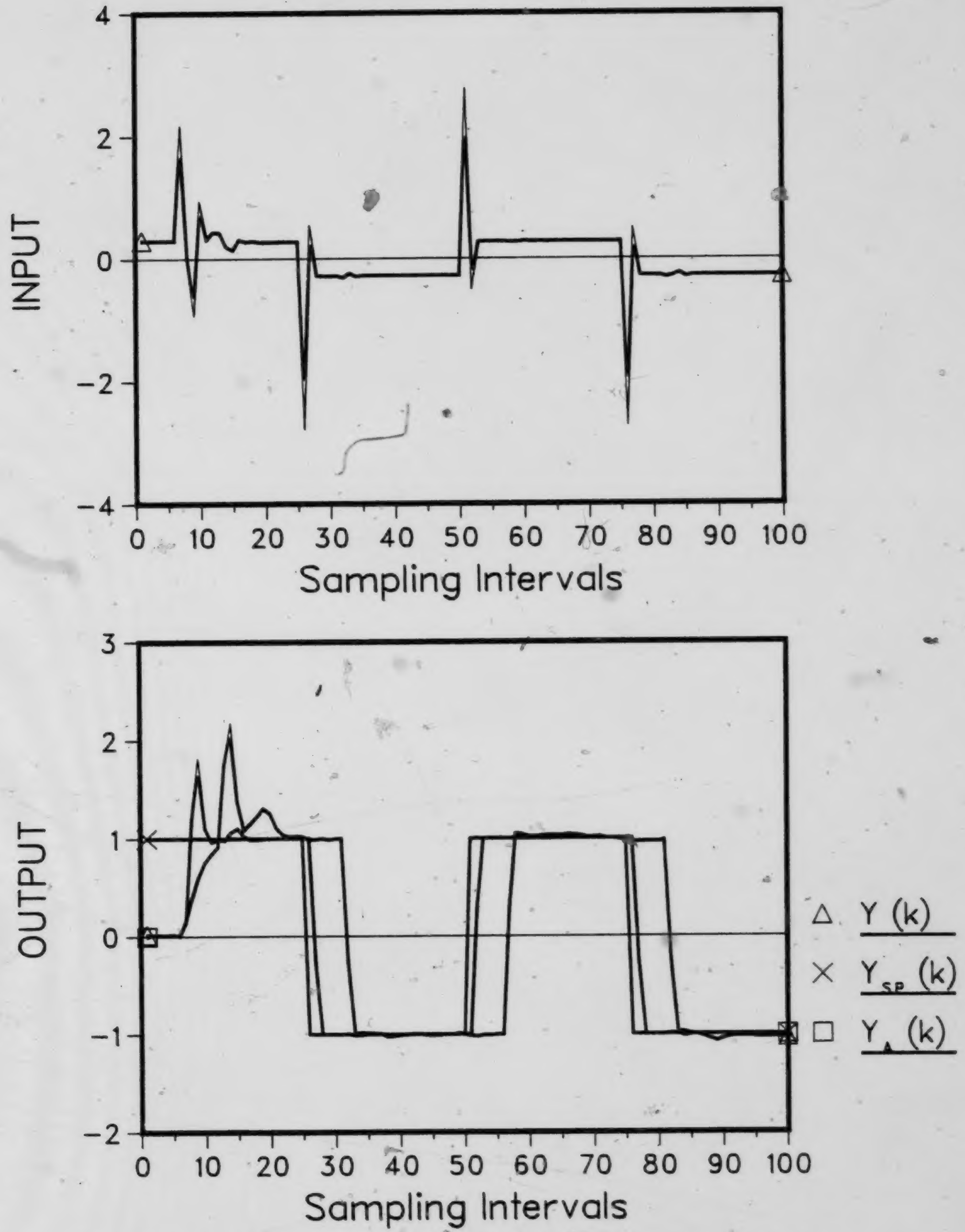


Fig.—4.9 Simulated process response
Ex 1/PASP/IA/SC.01/d=5/p=0./Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

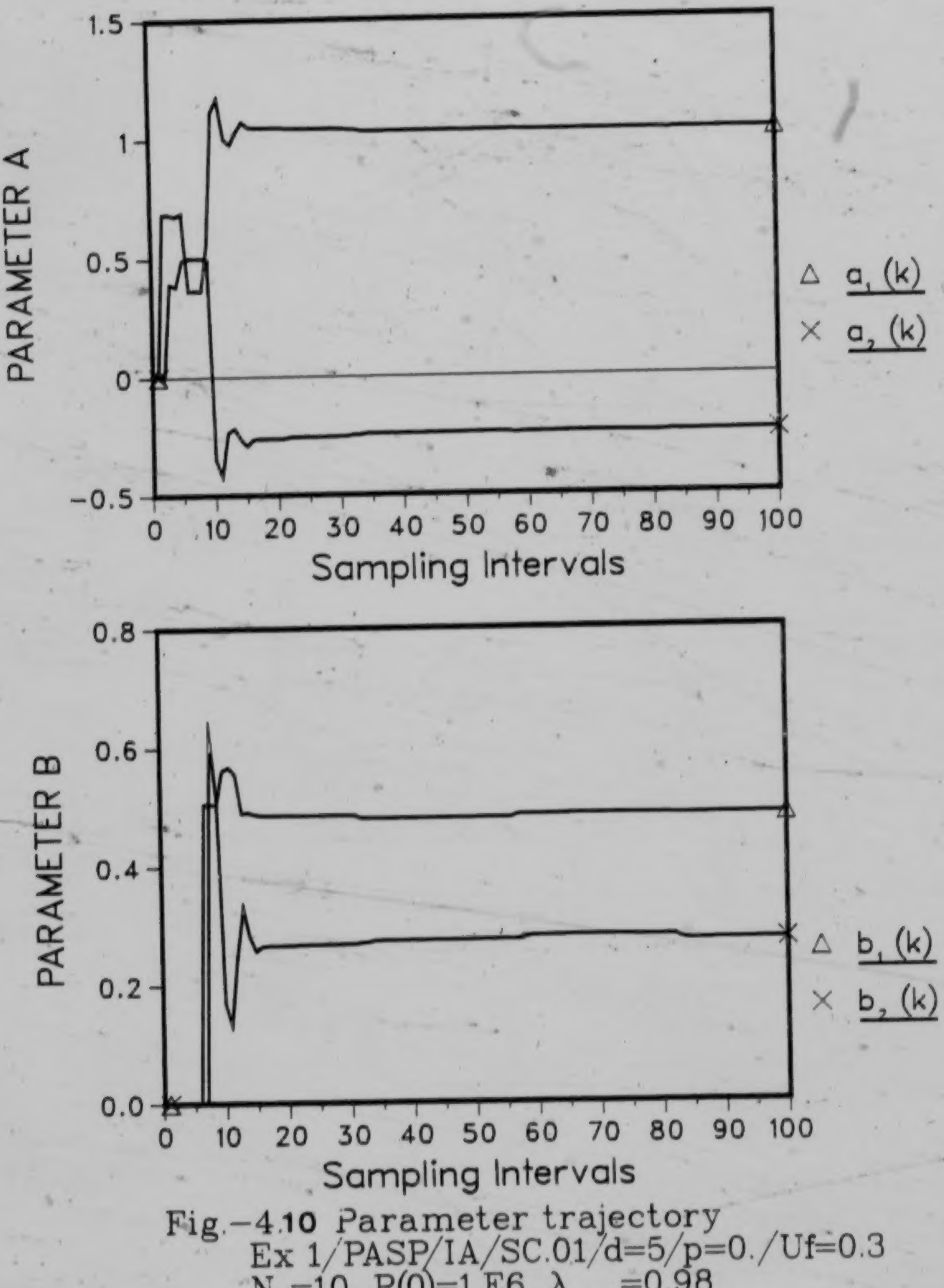


Fig.−4.10 Parameter trajectory
Ex 1/PASP/IA/SC.01/d=5/p=0./Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

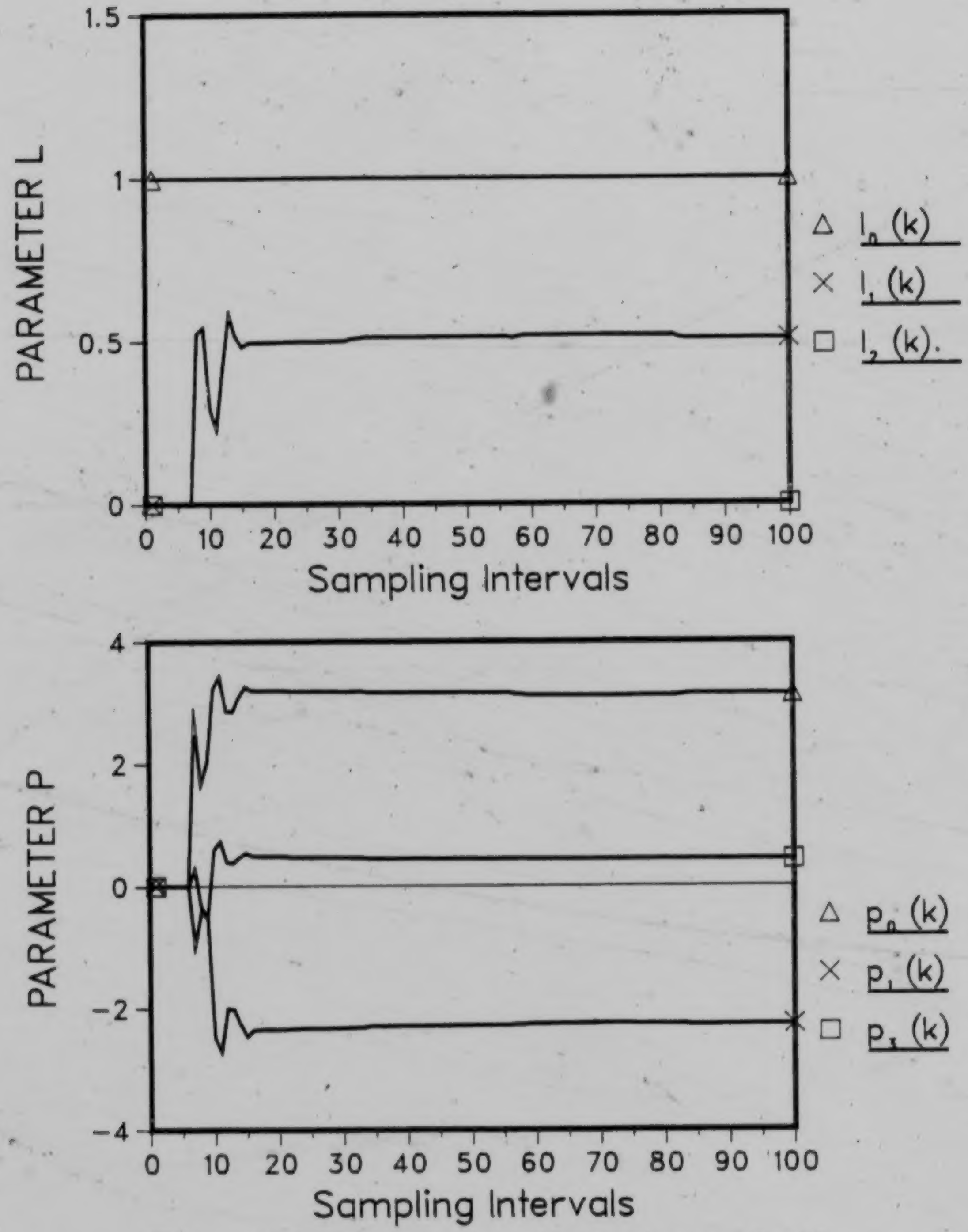


Fig.–4.11 Parameter trajectory of PA
Ex 1/PASP/IA/SC.01/d=5/p=0./Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

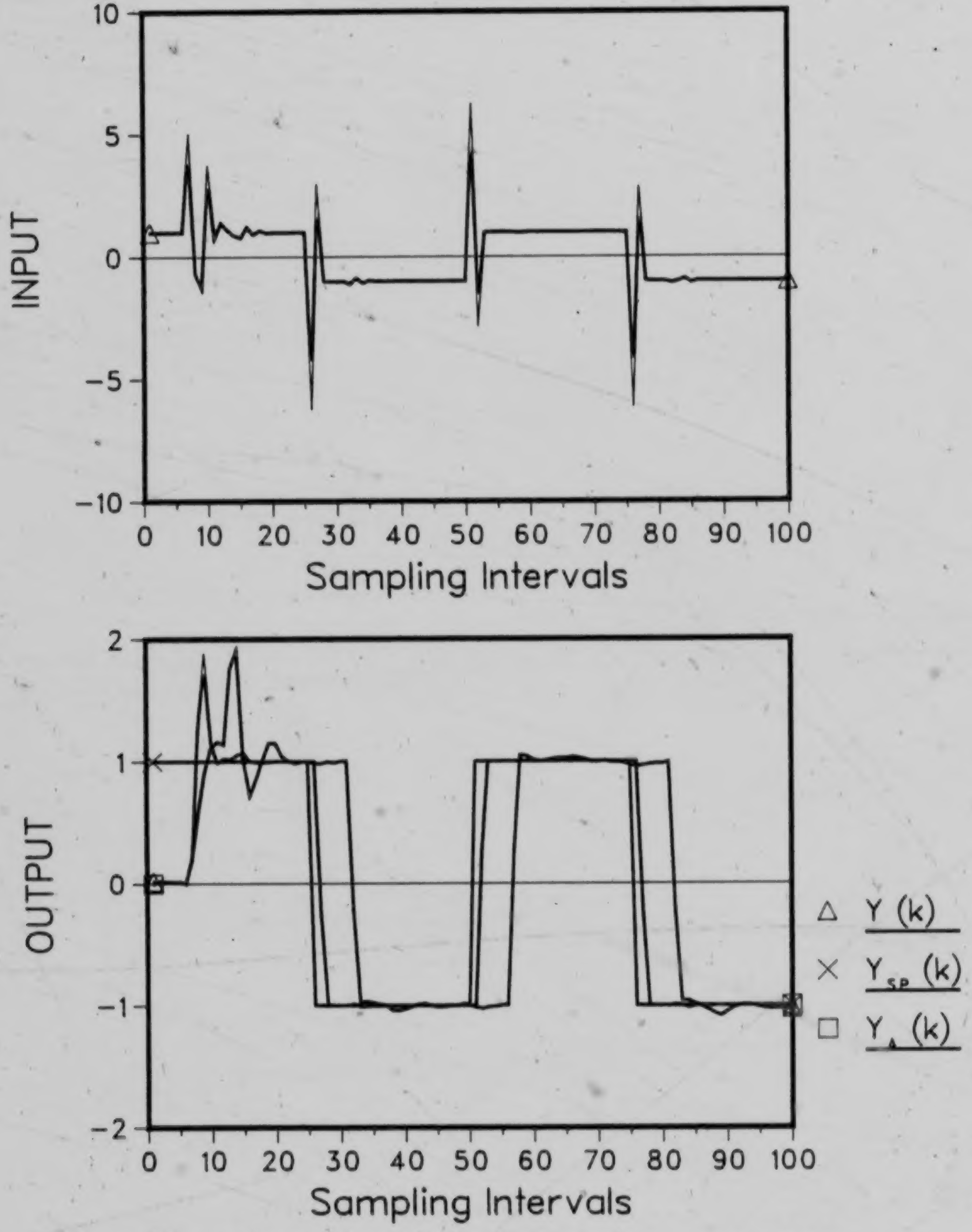


Fig.−4.12 Simulated process response
Ex 2/PASP/IA/SC.01/d=5/p=0./Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

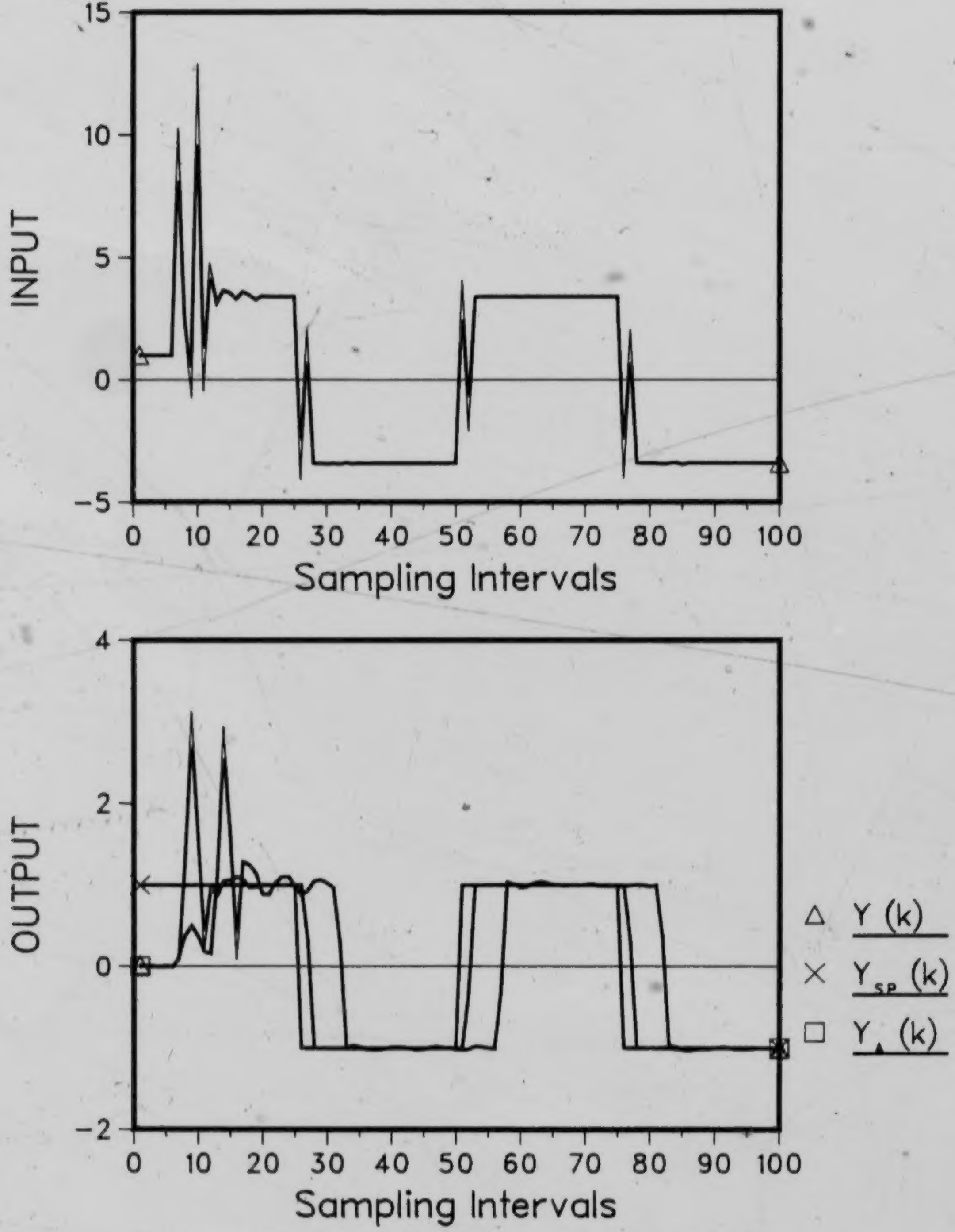


Fig.–4.13 Simulated process response
Ex 3/PASP/IA/SC.01/d=5/p=0./Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

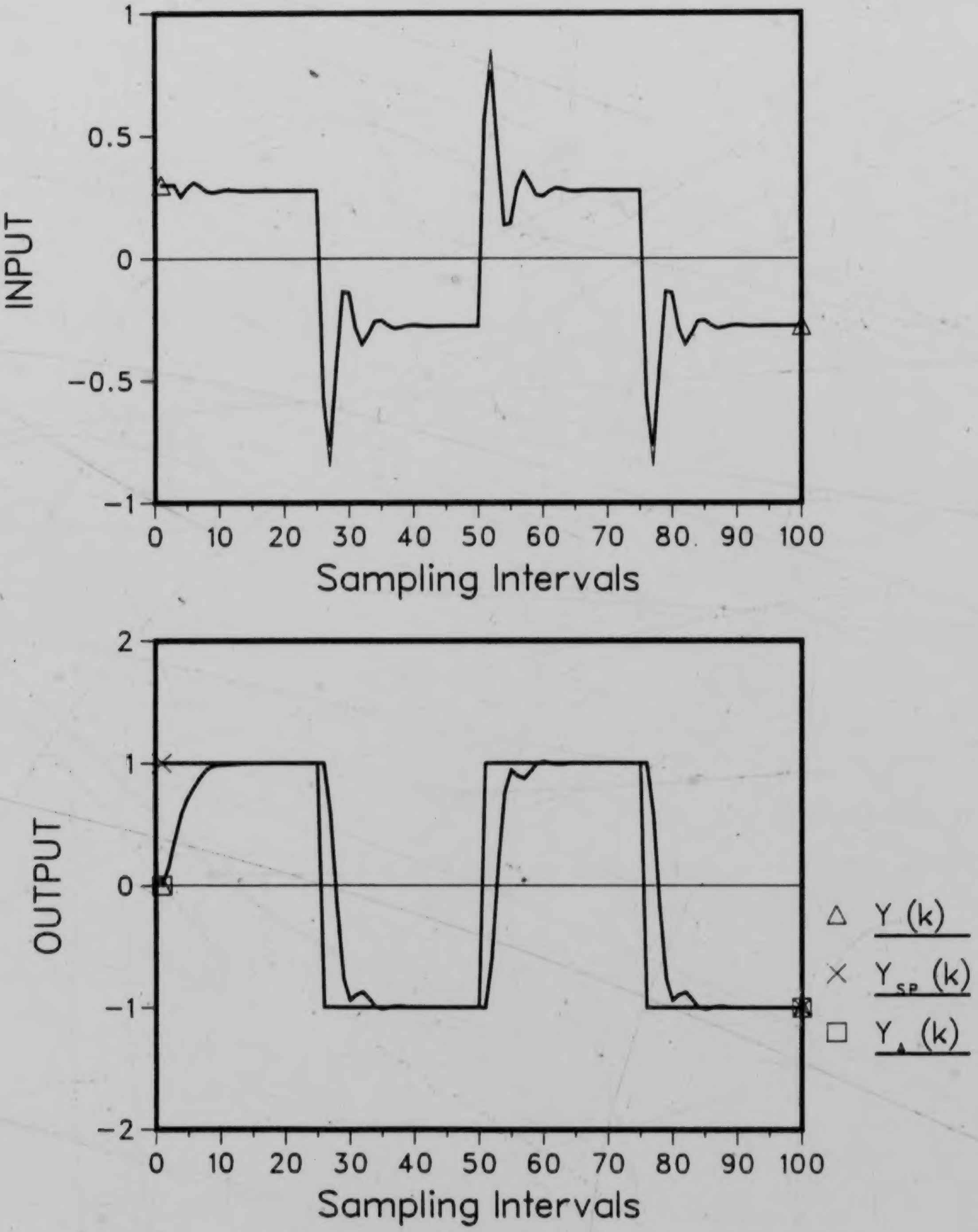


Fig.−4.**14** Simulated process response
Ex 1/PID/DC/d=0/p=0.535/Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

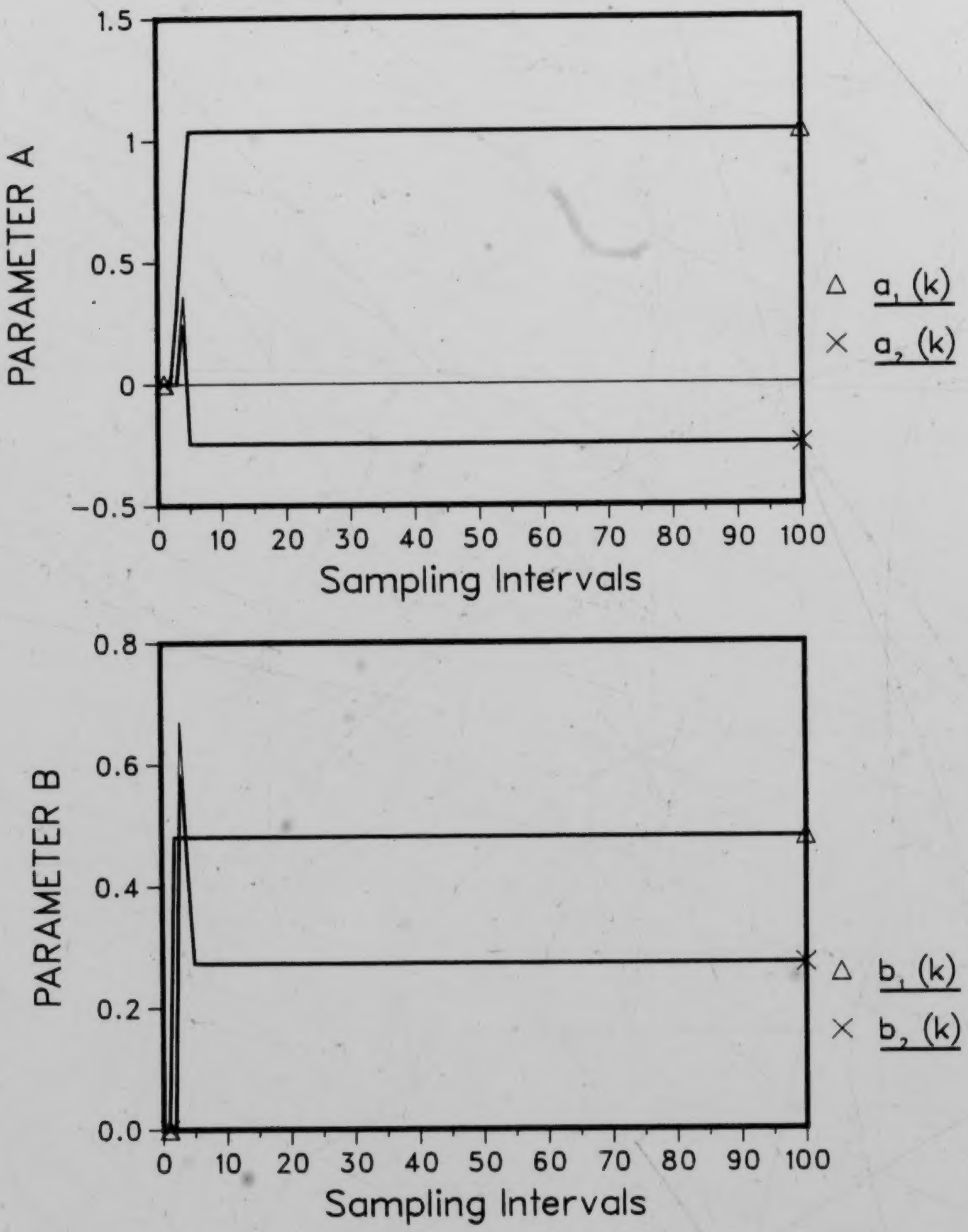


Fig.−4.15 Parameter trajectory
Ex 1/PID/DC/d=0/p=0.535/Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

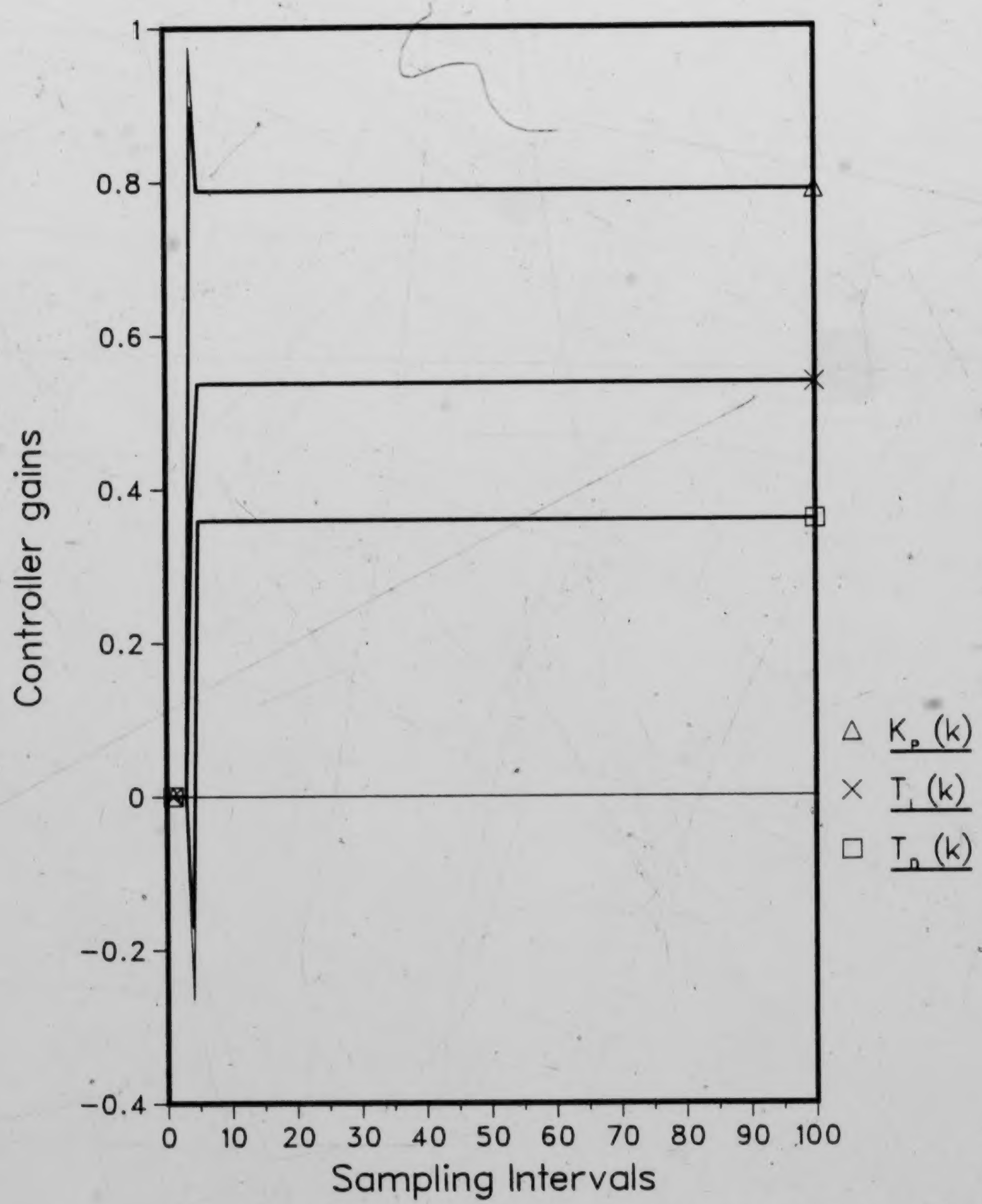


Fig.−4.16 Parameter trajectory of PID
Ex 1/PID/DC/d=0/p=0.535/Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

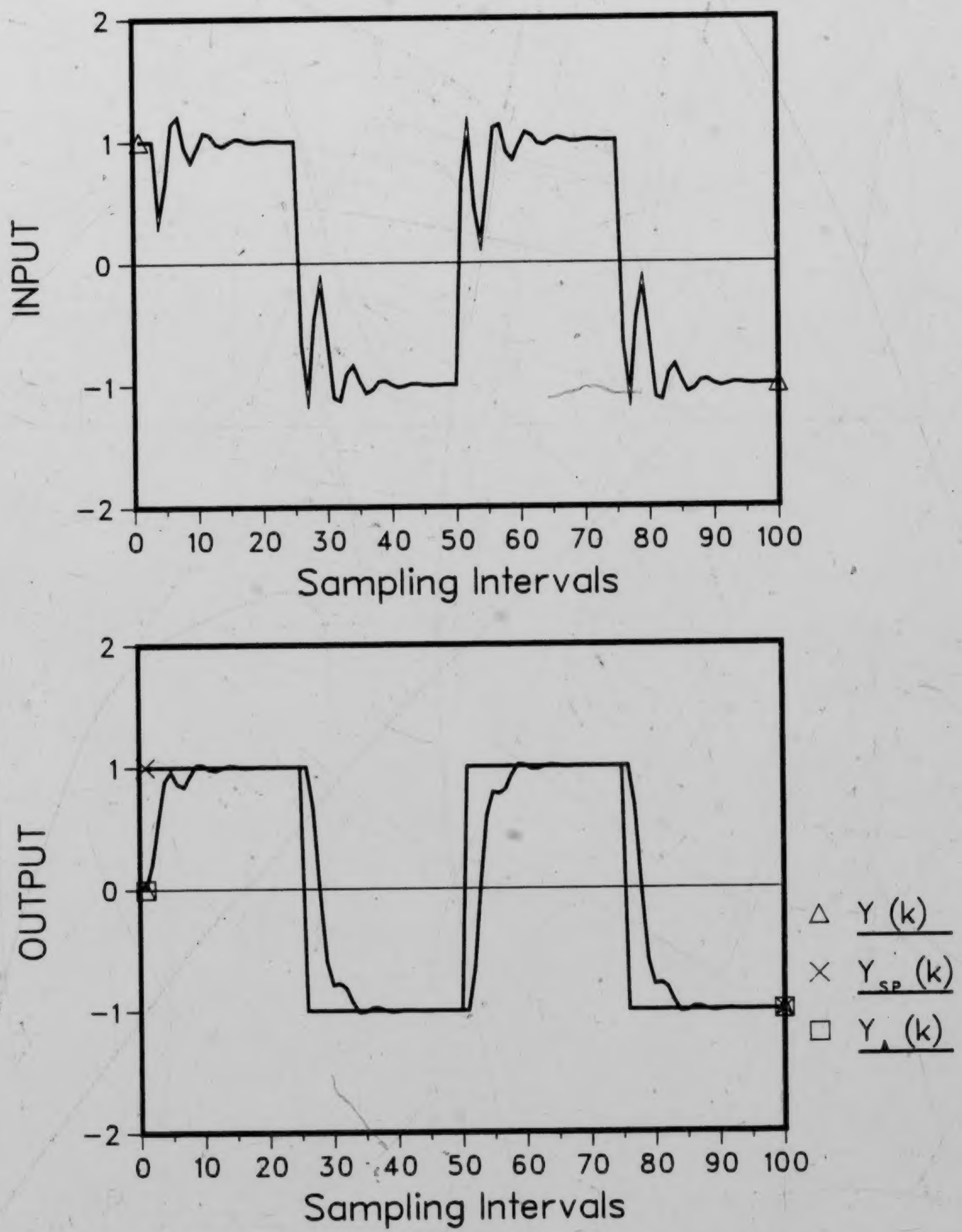


Fig.−4.17 Simulated process response
Ex 2/PID/DC/d=0/p=0.6/Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

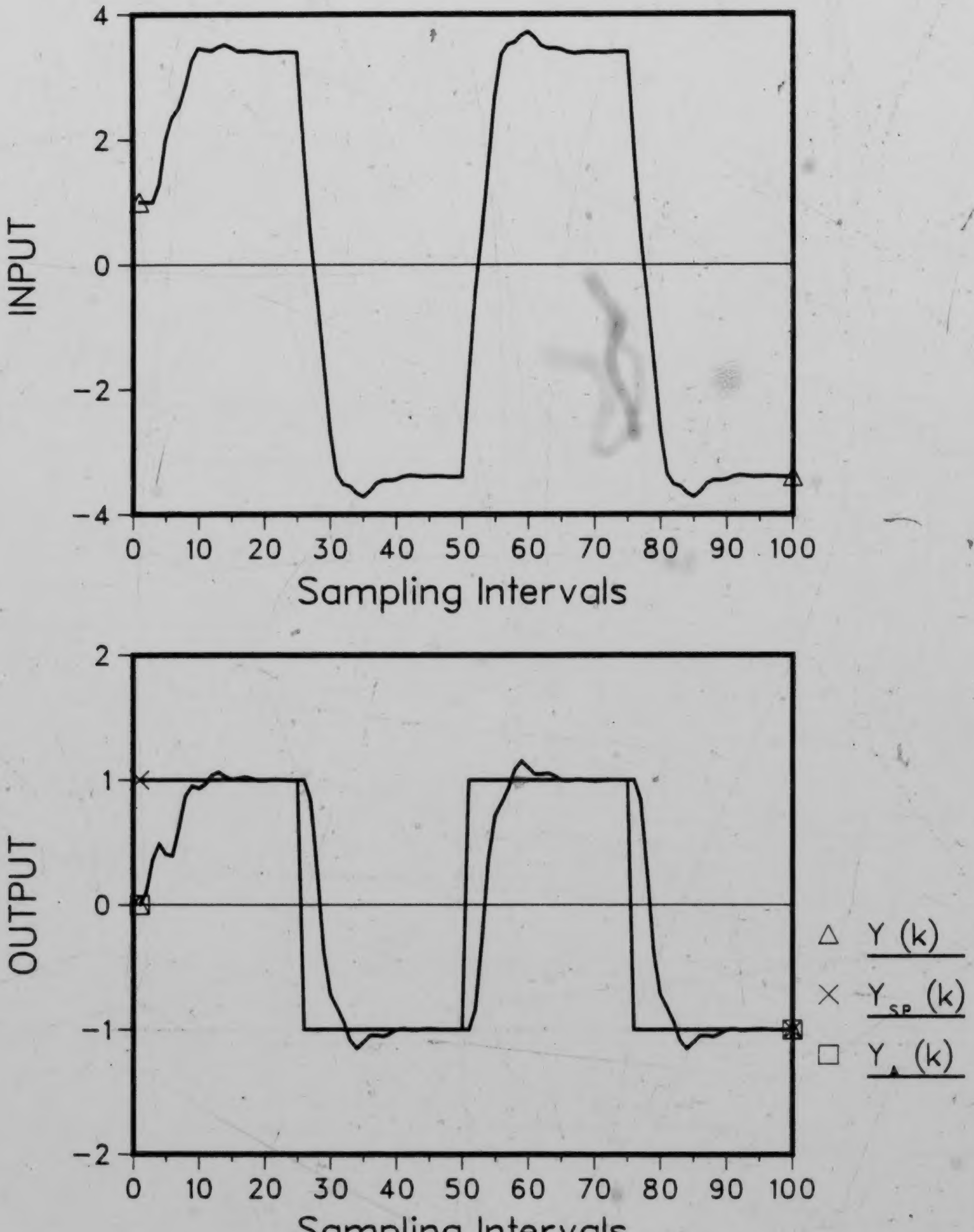


Fig.—4.**18** Simulated process response
Ex 3/PID/DC/d=0/p=0.55/Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

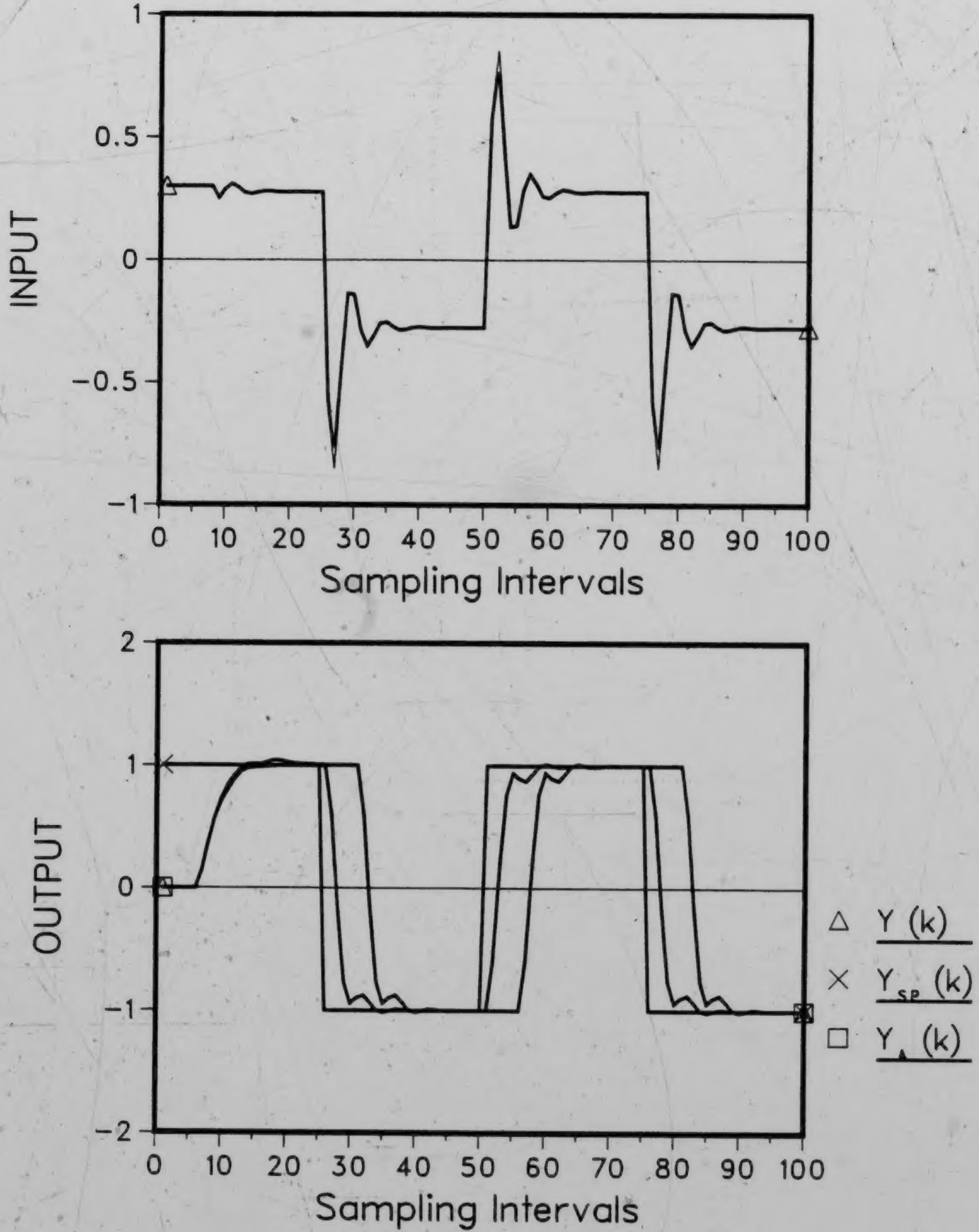


Fig.−4.**19** Simulated process response
Ex 1/PID/DC/d=5/p=0.535/Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

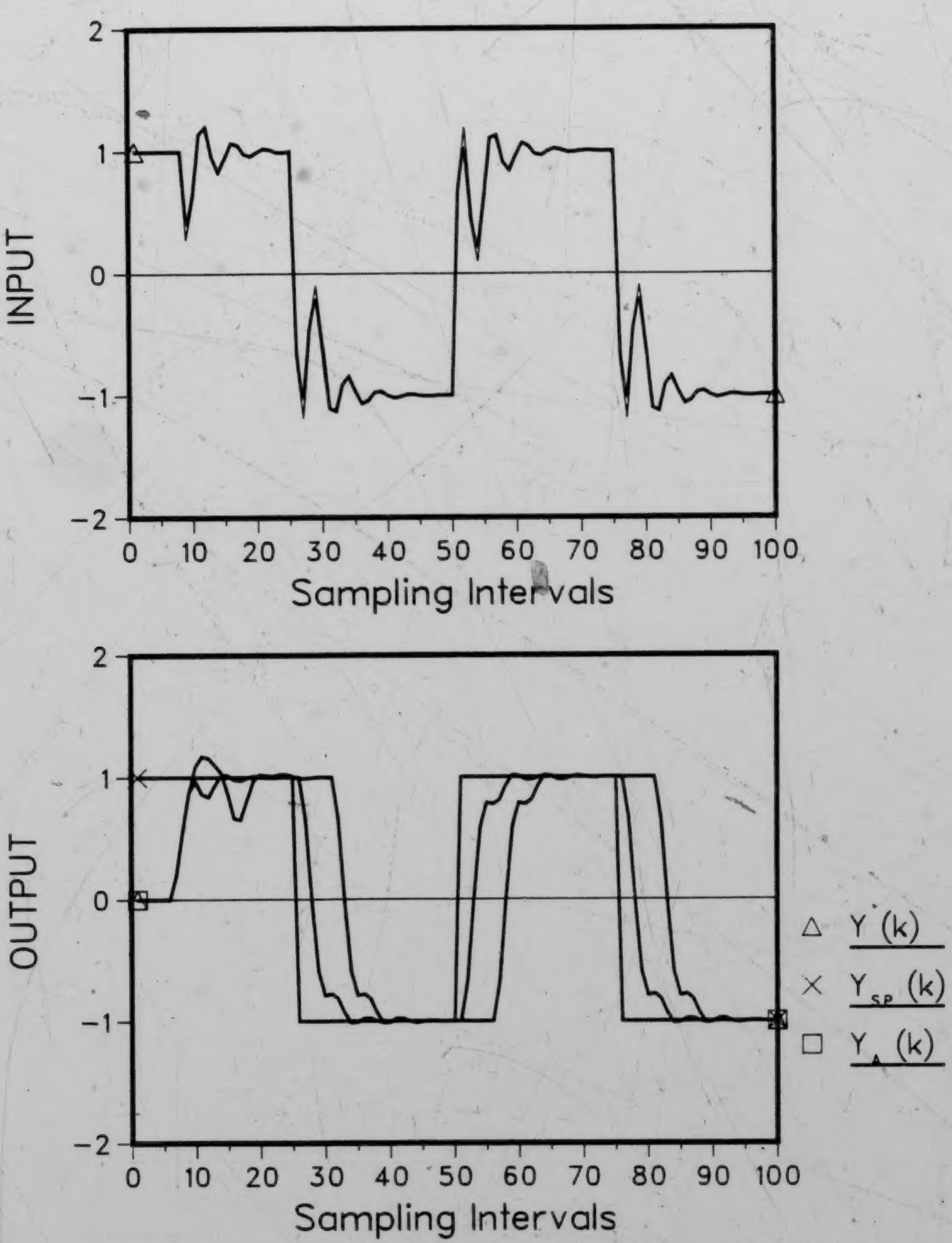


Fig.–4.**20** Simulated process response
Ex 2/PID/DC/d=5/p=0.6/Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

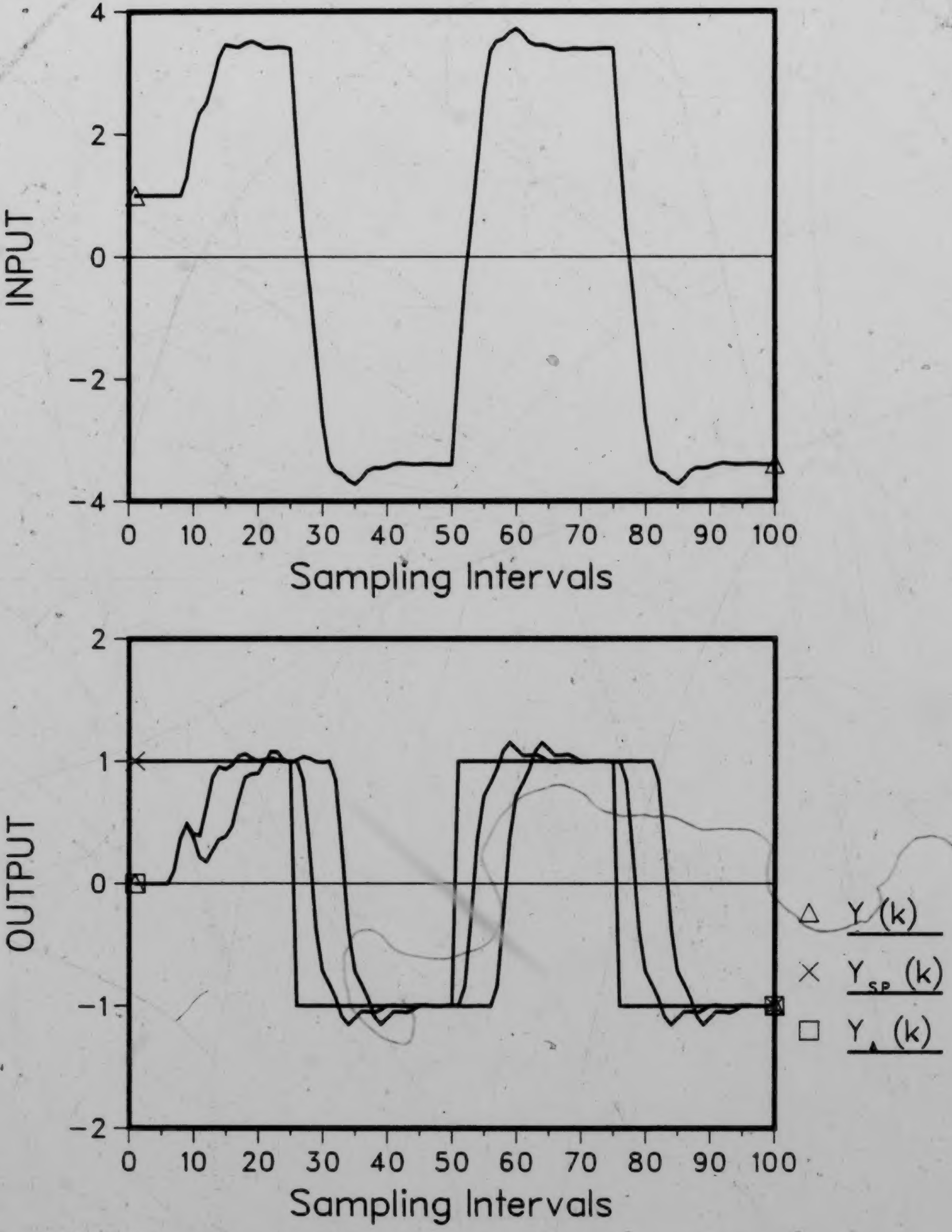


Fig.–4.**21** Simulated process response
Ex 3/PID/DC/d=5/p=0.55/Uf=1
N_o=10 P(0)=1.E6 λ_{max}=0.98

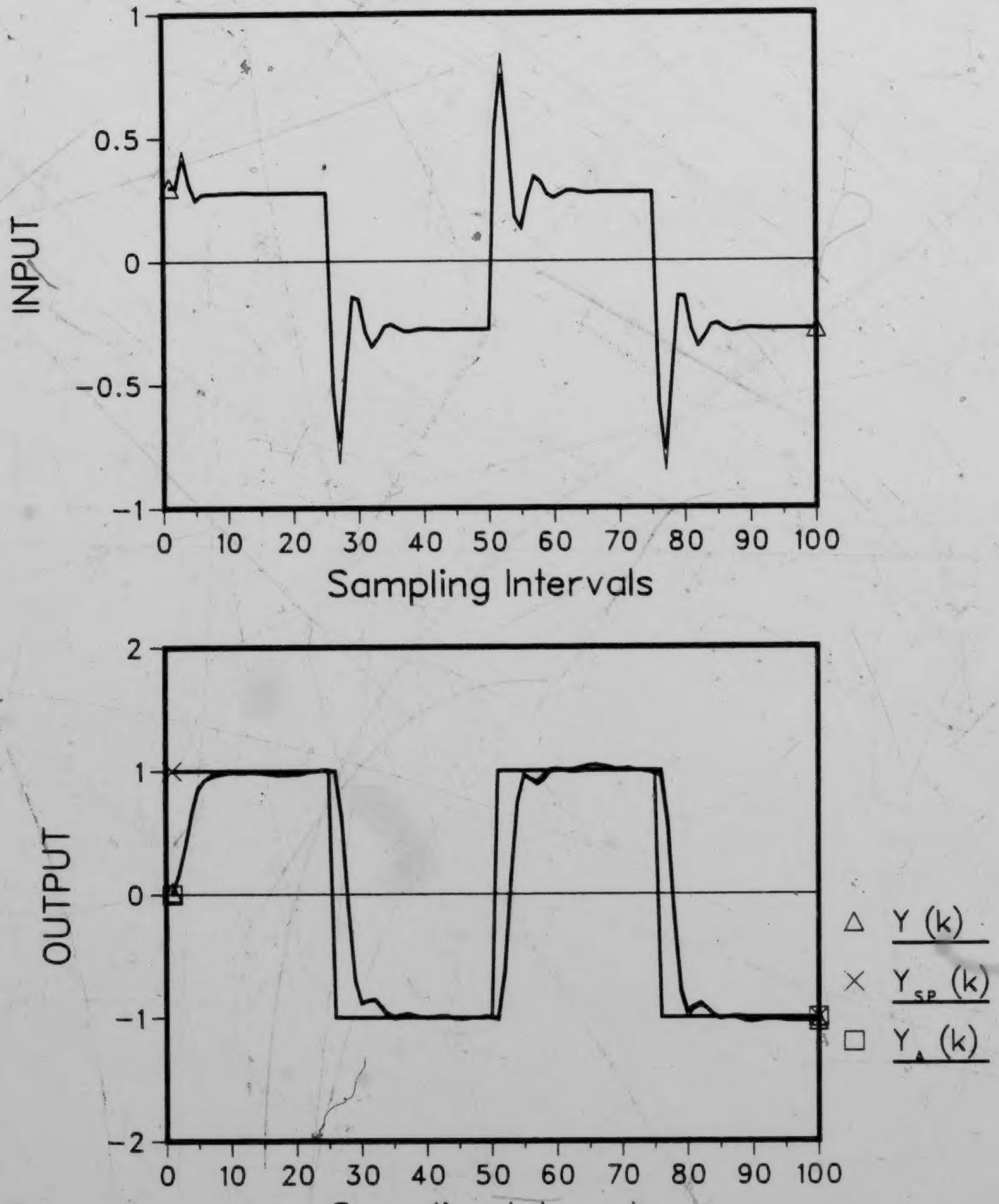


Fig.–4.**22** Simulated process response
Ex 1/PID/SC.01/d=5/p=0.535/Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

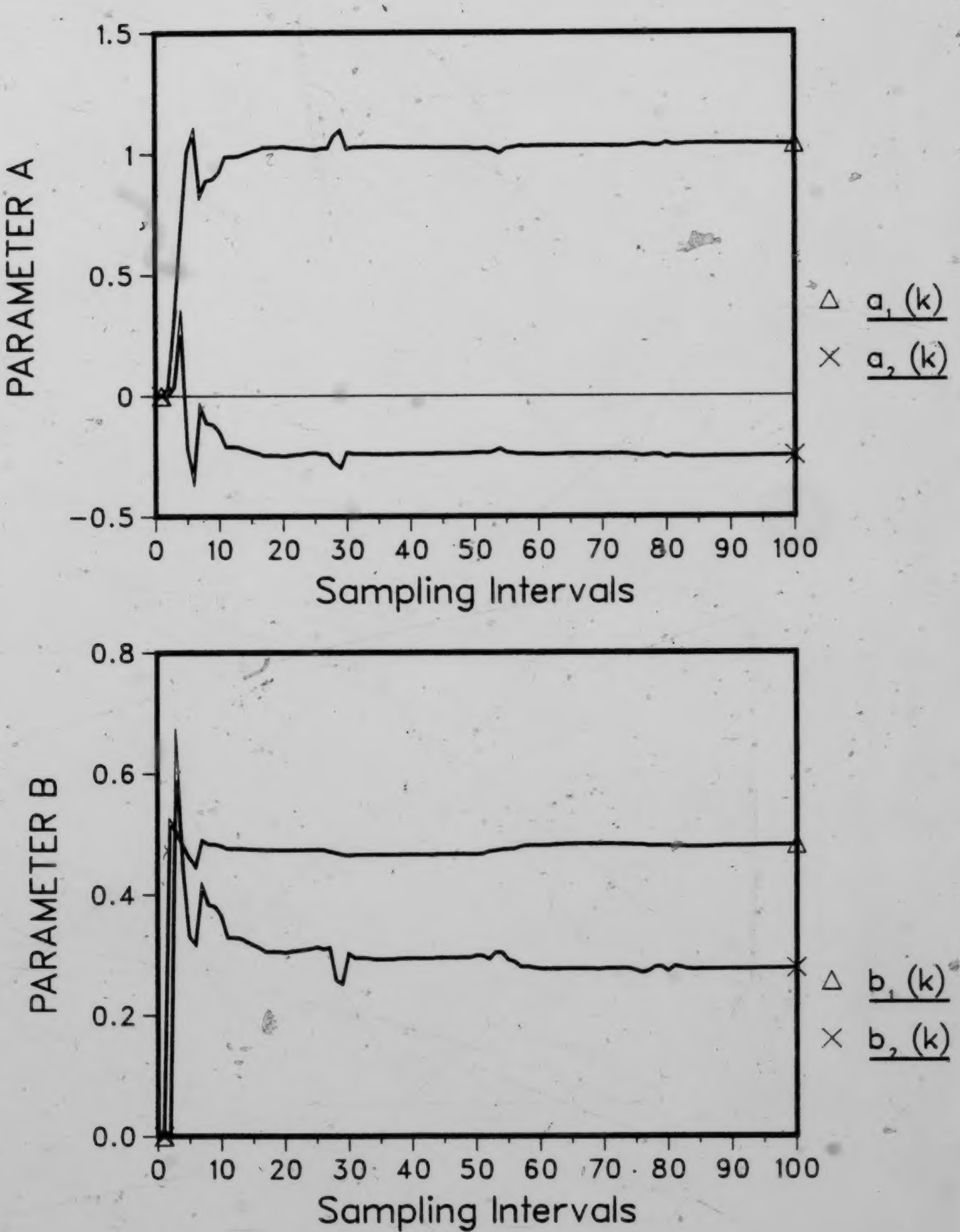


Fig.−4.**23** Parameter trajectory
Ex 1/PID/SC.01/d=0/p=0.535/Uf=0.3
N_0=10 P(0)=1.E6 λ_{max}=0.98

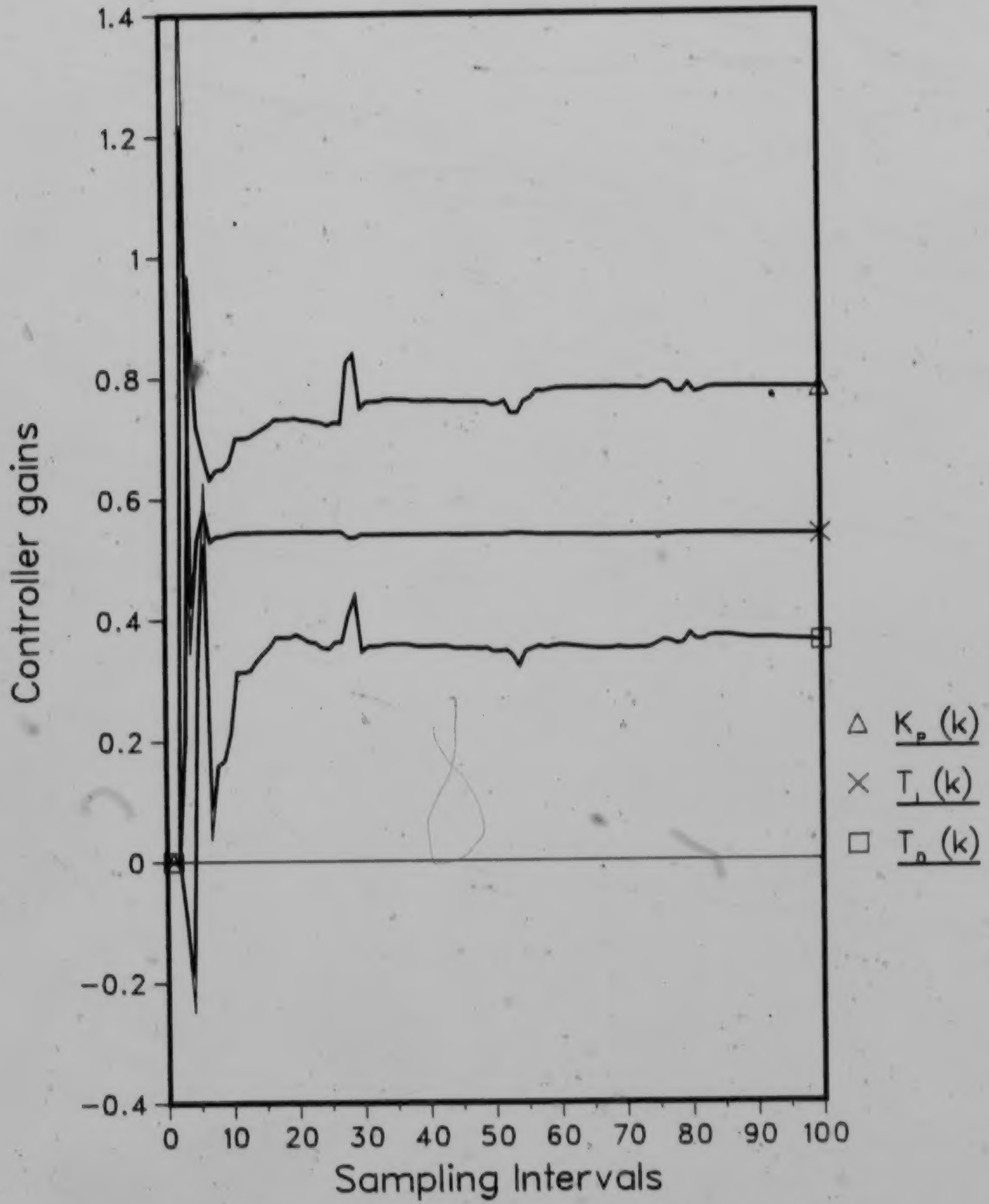


Fig.−4.24 Parameter trajectory of PID
Ex 1/PID/SC.01/d=5/p=0.535/Uf=0.3
N_o=10 P(0)=1.E6 λ_{max}=0.98

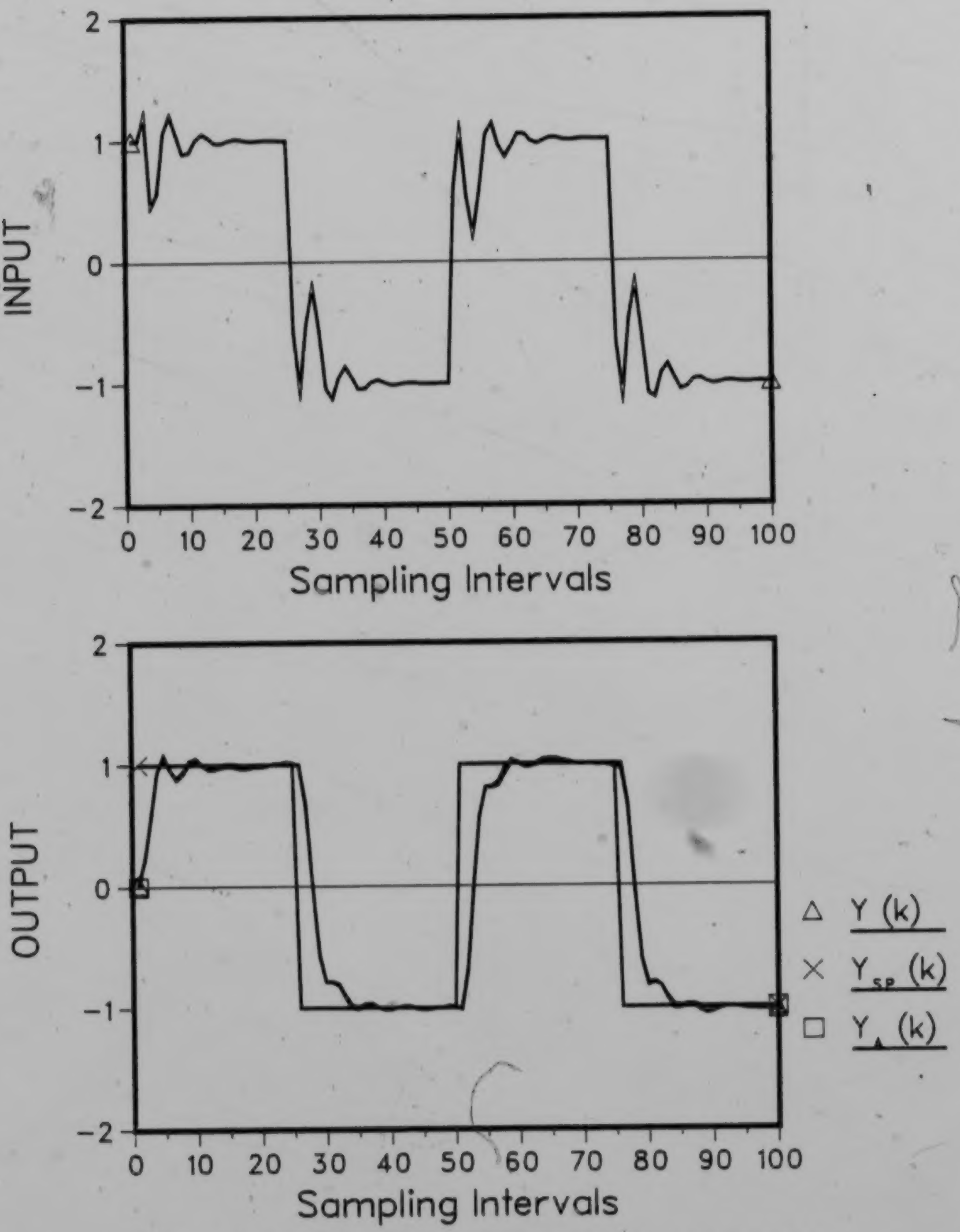


Fig.−4.25 Simulated process response
Ex 2/PID/SC.01/d=5/p=0.6/Uf=1.
N_0=10 P(0)=1.E6 λ_{max}=0.98

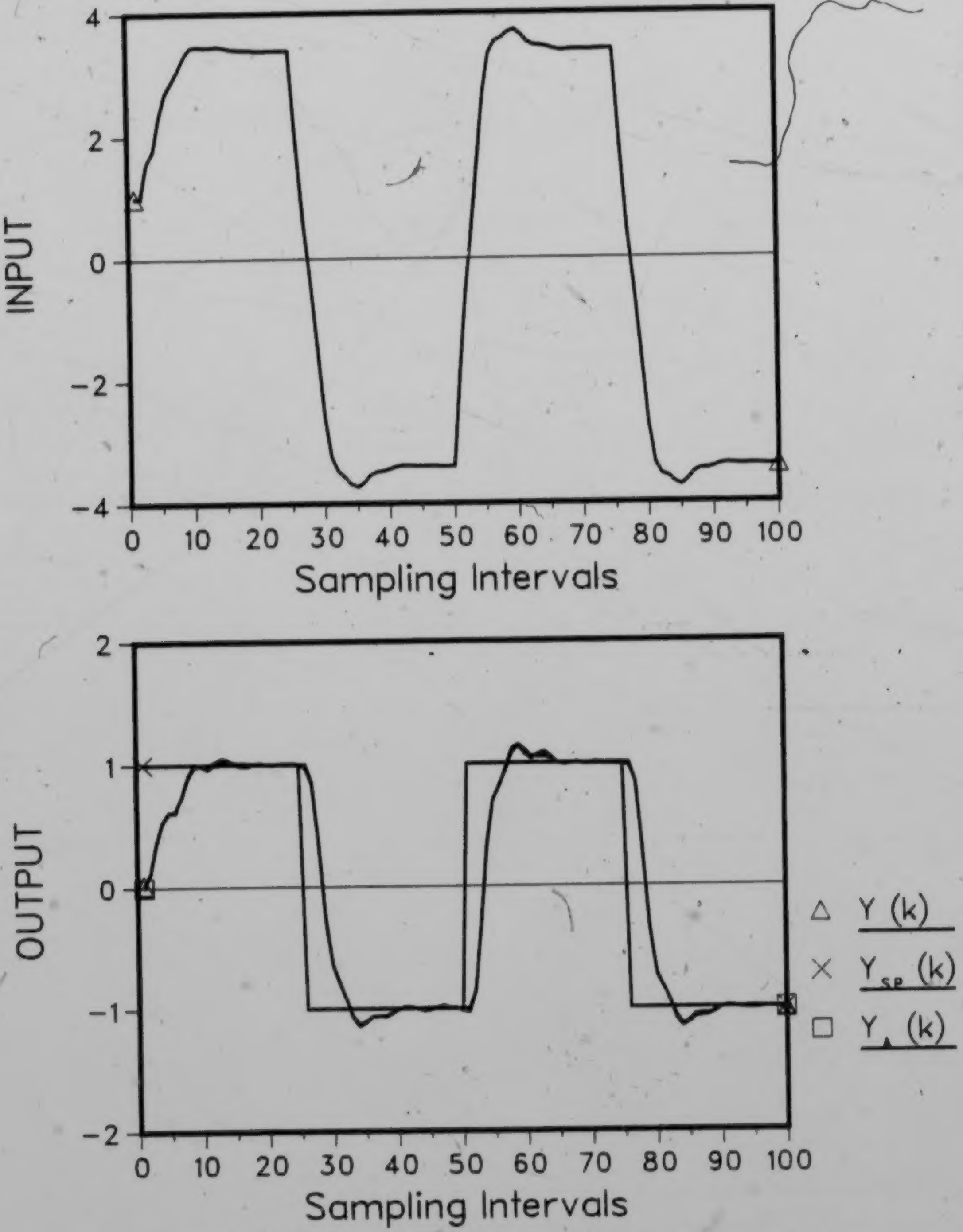


Fig.–4.**26** Simulated process response
Ex 3/PID/SC.01/d=0/p=0.55/Uf=1.
N_o=10 P(0)=1.E6 λ_{max}=0.98

4.5 Conclusions

In this chapter we used a Smith predictor to make the pole-assignment control algorithm discussed in the third chapter computationally more efficient for handling processes with large time delays. This control algorithm has been proven globally stable if a persistently exciting signal is used. It has also been shown that under certain assumptions this pole-assignment algorithm with the dead time compensation feature can take the form of a velocity type convetional three term adaptive PID controller for a particular class of processes. The excellent performance of these adaptive control algorithms has been illustrated by simulating minimum and nonminimum phase systems under different cases.

5. Evaluation of adaptive control strategies

5.1 Introduction

The simulation results presented in the previous chapters prove that the proposed pole-assignment control algorithm can easily handle minimum or nonminimum phase systems under different operating conditions. This motivates us to apply this control algorithm in a more complicated and difficult to control chemical process. The polymerization process of methylmethacrylate in a batch polymerizer is chosen as such a process. The batch solution polymerization reactor under investigation has highly nonlinear and time-varying characteristics.

Currently some one hundred million metric tons per year of synthetic polymers are produced in the world in a wide variety of polymerization reactors. But until recently "polymers were mainly manufactured in batch reactors from faithfully prepared recipes scaled up from the chemist's beaker" (Ray (1983)). So, there is a lot of incentive of improving our knowledge about the engineering aspects of the polymerization processes. Usually, someone is interested in operating the polymerization reactor in such a way that the final product has desirable properties while the operating cost is kept minimum. The benefits of automation and computer control of polymerization reactors as well as the state of the art of control in this area have been emphasized and reviewed by many authors (Amrehen (1977), Hoogendoorn and Shaw (1980), MacGregor et. al. (1983)). In most of the cases, optimal control theory has been applied and open loop control strategies have been derived to produce a polymer with desired final properties in an optimal manner (Hall (1960), Ray (1967), Osakawa and Fan (1970), Thomas (1984), Thomas and Kiparissides (1984) to name a few of the investigators). Although many optimal control strategies have been presented in literature, only a few of them have been applied experimentally (Chen et. al. ((1978), (1980), (1981), Ponnuswamy (1984)).

On the other hand there are some works where feedback control has been applied to polymer reactors. Keyes and Kennedy (1974) applied an adaptive control technique for the control of suspension PVC production. First, they developed a non-linear process model and secondly derived a regulator from the model utilizing variational techniques. Jo and Bankoff (1976) both experimentally and by simulation studied the solution polmerization of vinyl acetate in a CSTR, by using the Kalman filter. Kiparissides (1978) derived suboptimal stochastic control policies by solving a linear quadratic optimal control problem for a continuous latex reactor. Kiparissides and Shah (1983) evaluated two adaptive control algorithms and a fixed gain PID controller to a batch suspension PVC reactor. Despite the nonlinear and time-varying characteristics of the reactor, they obtained excellent control using either one of the two adaptive control strategies which performed better than the PID controller. This work has been extended by Cluett et. al. (1984) who used a globally stable Adaptive Predictive Control System (APCS) for setpoint tracking and constant rate control.

The small number of experimental applications of process control in the area of polymers has been attributed to the lack of development of on-line measuring devices. The viscous nature of the polymer mixture is the main reason for the difficulty of on-line measurements. It is anticipated that this problem will be solved soon and that the 1980's will see the application of advanced control strategies in the polymerization industry (MacGregor (1983)).

5.2 Experimental Reactor System

In this section, we shall briefly present the experimental system used in this work. For more details about this system, someone should be referred to Ponnuswamy (1984). A schematic representation of the experimental set-up is shown in Figure 5.1.

The reactor is a five liter jacketted cylindrical glass vessel. Its inside diameter is 152 mm while its height is 280 mm. The glass vessel is calibrated and marked to show the volume of the reaction mixture inside the reactor.

The mixing of the reactants is done by a stirrer motor assembly mounted on the reactor. The speed of the stirrer is kept constant throughout the course of polymerization by a controller which supplies the necessary power to the motor.

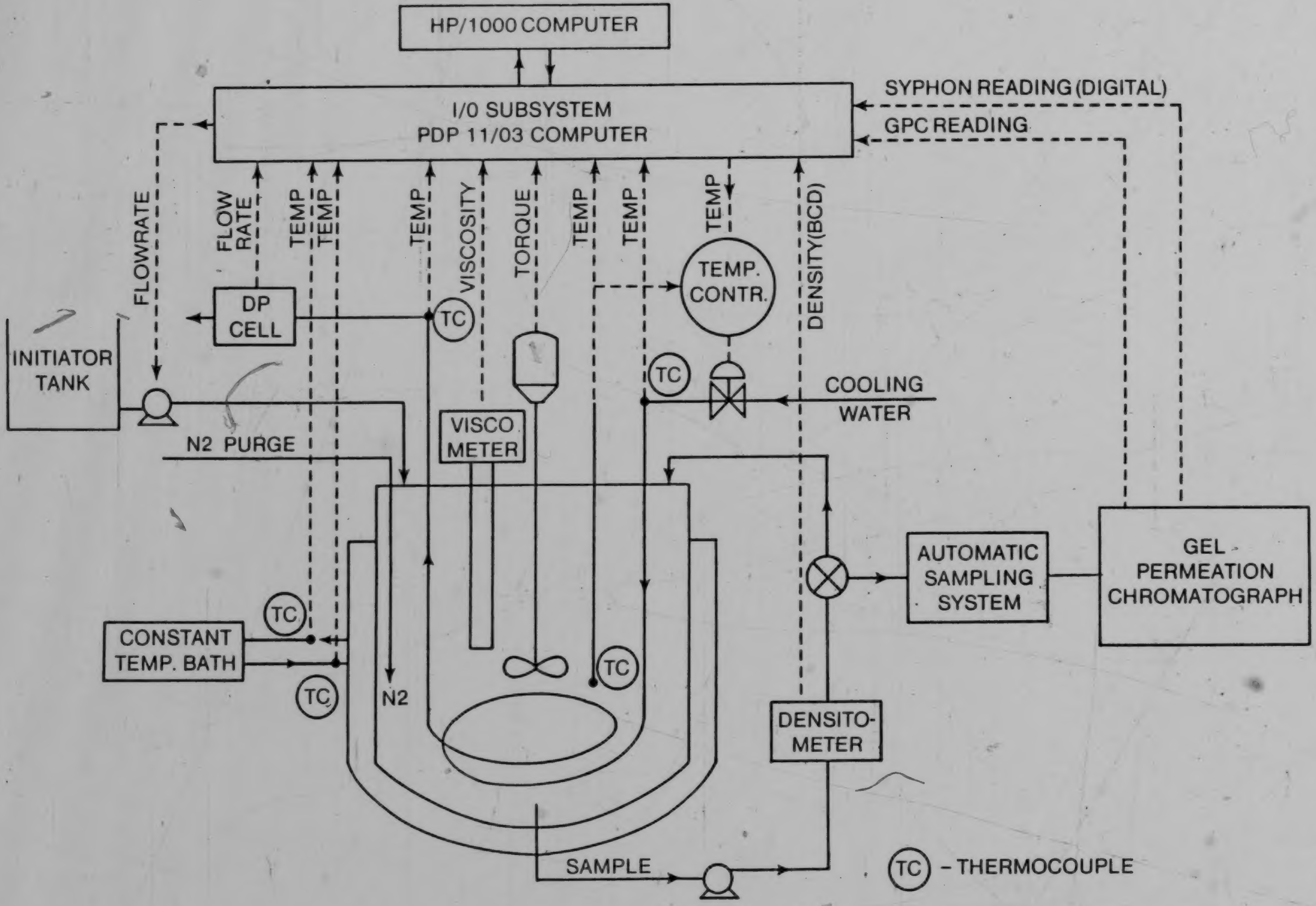

HP/1000 COMPUTER
I/0 SUBSYSTEM
PDP 11/03 COMPUTER
SYPHON READING (DIGITAL)
GPC READING
FLOWRATE
FLOW RATE
TEMP
TEMP
TEMP
VISCOSITY
TORQUE
TEMP
TEMP
TEMP
DENSITY(BCD)
DP CELL
TC
TEMP. CONTR.
INITIATOR TANK
TC
COOLING WATER
N2 PURGE
VISCO METER
GEL PERMEATION CHROMATOGRAPH
AUTOMATIC SAMPLING SYSTEM
TC
CONSTANT TEMP. BATH
TC
TC
N2
TC
DENSITO-METER
SAMPLE
TC - THERMOCOUPLE

FIG. 5.1 SCHEMATIC DIAGRAM OF POLYMERIZATION REACTOR SYSTEM.

During the course of polymerization the load on the stirrer increases because of the increasing viscosity. The torque on the stirrer is measured as a 0-10V output signal and can be used as an indication of the conversion level within the reactor.

The necessary heating is continuously supplied to the reaction mixture at a constant rate. Hot water is used for this purpose and is circulated through the jacket of the reactor by using a thermally protected pump. The temperature of hot water is measured at its inlet and outlet points at the reactor jacket. The constant temperature hot water tank has a capacity of 25 liters while the agitators within it help to keep a uniform temperature inside.

Since the reaction is exothermic, cooling of the reaction mixture is needed during some stages of the polymerization process. This is achieved by using cold water. The cooling water flows through a stainless steel coil provided inside the reactor. The inlet and outlet temperatures of the cooling water are measured using thermocouples. Its flowrate is adjusted by a Foxboro feedback controller and is used as the control variable in the control studies.

The viscosity of the reaction mixture is measured by a viscometer mounted on the top of the reactor. The operation of this viscometer is based on a falling piston principle. During its operation, the piston is raised by a lifting mechanism while reaction mixture occupies the space formed below the piston rod assembly. The piston is then allowed to fall while its falling time is an indication of the reaction mixture viscosity. Measurements of the viscosity can be used in conversion calculations using suitable correlations.

During the course of polymerization, the reaction mixture is continuously circulated through a densitometer. A reciprocating-revolving pump suitable for handling highly viscous fluids is used to circulate the reaction mixture. Teflon tubing is used to connect the pump with the reactor. The temperature effect on the density measurements is eliminated by performing them at a constant temperature. Because of it, the reaction mixture is first passed through a thermostatic bath before entering the densitometer. The accuracy of the densitometer is 0.0001 gr/cm^3 while measurements can be taken every five seconds. Since the density of the reaction mixture increases during the course of polymerization, it can be used as an indication of the conversion level within the reactor. In this work, a suitable correlation between density and conversion has been developed. So, the densitometer is used as an on-line measuring device for conversion control purposes.

In the experimental set-up, a Foxboro PID controller is also included. This controller is used in this work as a slave controller in a cascade control configuration. It receives a remote setpoint from the computer (which is created by the master controller). This signal is compared with the temperature within the reactor and the appropriate control signal (10-50 ma) is sent to an electro-pneumatic converter. The output of this converter (3-15 psi) is sent to a diaphragm control valve to regulate the cold water flowrate.

After briefly describing the experimental system, the correlation between density and conversion will be presented in the next section.

5.3 Conversion Measurements

As it was previously mentioned, a number of control strategies have been applied to control useful properties of the polymer during the course of polymerization. But it was made clear that only few of them have been tested experimentally. The main reason for this is the difficulty for getting on-line measurements because of the viscous nature of the reaction mixture.

In this work we intend to experimentally control the conversion trajectory within a batch polymerization reactor of methylmethacrylate by applying feedback control. So, it is necessary to have on-line measurements of the conversion. Before we present our way of getting on-line measurements of the conversion we shall mention a number of techniques used to estimate conversion.

Between the methods available to measure the conversion the most direct method is to stop the polymerization, isolate and weigh the polymer. Obviously this is not useful for on-line conversion measurements.

Since the polymer has greater density than the monomer, the reaction volume decreases as the polymerization process proceeds. If a linear relationship between conversion and reaction volume is assumed, it will be easy to calculate conversion from the volume decrease. Still, this is not a useful method since in practice it is not easy to measure the exact reaction mixture volume.

Another method of measuring conversion is based on measurements of the refractive index. In this method we exploit the difference between the structures of the polymer and monomer because of rearrangement of chemical bonds.

A more accurate method is the gas chromatography method. In this method we calculate the mass fraction of the volatile components due to monomer with respect to the volatile components due to an internal standard which is the solvent. Despite its advantages, there is a number of disadvantages associated with this method (Berezkin et. al. (1977)). This method can be used for on-line measurements if an automatic sampling procedure is available.

Since each method of measuring conversion had its advantages and disadvantages, it was considered more practical to develop a correlation between the conversion and a process variable that is easily on-line measured. Torque on the stirrer, viscosity of the reaction mixture or density of it could be used as such process variables. In the experimental system used, all these variables are measured. Measurements of the torque were not considered as reliable because of the friction on the bearing site of the stirrer. Between the other two alternatives, we decided to use the density since the densitometer used gave fast and accurate measurements. We need fast measurements to avoid time delays in the closed loop control system.

There are two ways of correlating conversion and density. In one of them, mathematical equations relating density and conversion of the polymerization mixture are developed based on the assumption of ideal mixing of polymer, monomer and solvent. This way was used by Abbey (1981) and Schmidt and Ray (1981) to correlate conversion and density in the case of emulsion polymerization of methylmethacrylate. The second way involves the development of an empirical correlation between conversion and density. This way will be used in this work.

In order to develop this correlation, we run the reactor under PID control while the following empirical correlation

$$x = 961.78\,\rho - 827.52$$

was used. In this equation, the density of the reaction mixture ρ is in gr/cm^3 while the conversion x varies from 0 to 1. During this run, density measurements were taking on-line by operating the densitometer at a constant temperature of 60 degrees Celsius. The corresponding conversion values were calculated by using the gas chromotograph off-line. These measurements are reported on Table 5.1.

Based on the data of Table 5.1 an improved correlation between conversion and density was developed by applying the least squares method. This correlation is given by the following equation

$$x = 1021.756\,\rho - 884.587$$

In practice we found that the correlation given

TABLE 5.1

Density gr/cm^3	Conversion
0.8700	3.89
0.8777	12.95
0.8917	25.02
0.9044	39.57
0.9112	45.90
0.9204	59.93
0.9284	63.64
0.9356	69.30

by the following equation

$$x = 1021.756\,\rho - 880.587$$

describes better the polymerization process. This correlation will be used in all the experimental runs of this work.

So far, we have presented different methods of measuring conversion in a polymerization process and developed an empirical correlation between conversion and density for the solution polymerization of methylmethacrylate.

For simulation studies purposes, a mathematical model of the polymerization process must be available. In the next section, a mathematical model describing the polymerization of methylmethacrylate will be presented.

5.4 Kinetic Model

Polymers can be classified into two classifications. One of them divides polymers into condensation and addition polymers on the basis of the compositional difference between the polymer and the monomer(s) from which it was synthesized. The other classification divides them into step and chain polymers on the basis of the mechanism of the polymerization reactions.

The polymer methylmethacrylate is classified as a chain polymer. Chain polymerization is initiated by some reactive species produced from some compound called initiator. The reactive species adds to a monomer molecule to form a new reactive center. The process is repeated to continuously propagate the reactive center. During the course of polymerization, monomer, high polymer and growing polymer are present in the reaction mixture. In this type of polymerization, polymer of high molecular weight is formed immediately and remains relatively unchanged during the course of the process. This which only depends on time is the amount of polymer formed. This is the difference of chain polymerization from step polymerization where both the molecular weight and the amount of polymer are time dependent.

The free radical polymerization of methylmethacrylate is a very well known process. Several investigators (Matheson et. al. (1949), Hayden and Melville (1960), Balke (1972), Mahabadi and Meyerhoff (1979), Schmidt and Ray (1981), Ponnuswamy (1984) to name a few) have studied the kinetics of this process. In this work, the mathematical model describing the kinetics of this process and presented by Ponnuswamy (1984) will be used. This model will be briefly considered in the sequel.

The polymerization process is consisting of three steps: initiation, propagation and termination. The initiation step involves the dissociation of catalyst into two reactive species and the addition of one radical to a monomer molecule to produce a chain. Propogation consists of the growth of this chain by the successive additions of monomer molecules. Termination can also take place by disproportionation when transfer of a hydrogen radical, which is beta to one radical center, to another radical center takes place. The result is the formation of one saturated and one unsaturated polymer molecule. Finally, termination can take place if a polymer radical reacts with solvent or monomer molecules. These reactions are called transfer to solvent and transfer to monomer reactions respectively.

The general description of the chemical reactions for the polymerization of methylmethacrylate initiated by benzoylperoxide is as follows:

Initiator decomposition

$$I \xrightarrow{k_d} 2\ f\ R\cdot$$

Initiation

$$R\cdot + M \xrightarrow{k_i} R_1\cdot$$

Propogation

$$R_x\cdot + M \xrightarrow{kp} R_{x+1}\cdot$$

Transfer to monomer

$$R_x\cdot + M \xrightarrow{k_m} P_x + R_1\cdot$$

Transfer to solvent

$$R_x\cdot + S \xrightarrow{k_s} P_x + R_1\cdot$$

Termination by disproportionation

$$R_x\cdot + R_y\cdot \xrightarrow{k_{td}} P_x + P_y$$

Termination by compination

$$R_x\cdot + R_y\cdot \xrightarrow{k_tc} P_{x+y}$$

where I denotes initiator molecules, R· the initiator radicals, M the monomer molecules, $R_x\cdot$ and $R_y\cdot$ the live radicals of chain length x and y respectively S the solvent molecule and P_x and P_y dead polymer molecules of chain length x and y respectively. The parameters kd, ki, kp, km, k_s,k_{td} and k_{ts} are the rate constants of the corresponding reactions. The initiator efficiency is denoted by f. This parameter is defined as the fraction of the radicals produced in the dissociation reaction which initiate polymer chains. Its value is always less than one and varies between 0.3 and 0.8 (Odian (1981)).

Under the following assumptions

A1] All the reactions are irreversible.

A2] Reaction rate constants are independent of chain length.

A3] The concentration of radicals quickly becomes and remains constant during the course of polymerization (quassi steady state approximation)

A4] The monomer consumption due to initiation step is negligible.

A5] There are no temperature gradients within the reactor.

A6] The degree of mixing is infinity

and some order of magnitude studies (Thomas (1981)), the following simplified kinetic model results:

$$\frac{d[I]}{dt} = - k_d [I] \tag{5.1}$$

$$\frac{dx}{dt} = k_1 (1-x) [I]^{0.5} \tag{5.2}$$

$$\frac{d\mu_0}{dt} = Cf\, k_d [I] + K_2 [M] [I]^{0.5} + K_3 [I]^{0.5} \tag{5.3}$$

$$\frac{d\mu_2}{dt} = K_4 S [M]^2 [I]^{-0.5} + K_5 [M]^3 [I]^{-0.5} + K_6 [M]^2 \tag{5.4}$$

where

$$K_1 = kp (2fk_d/k_t)^{0.5} = A_1 \exp(-E_1/RT) \tag{5.5}$$

$$K_2 = k_m (2fk_d/k_t)^{0.5} = A_2 \exp(-E_2/RT) \tag{5.6}$$

$$K_3 = k_s (2fk_d/k_t)^{0.5} = A_3 \exp(-E_3/RT) \tag{5.7}$$

$$K_4 = \frac{2kp^2 k_s}{(2fk_d k_t{}^3)^{0.5}} = A_4 \exp(-E_4/RT) \qquad (5.8)$$

$$K_5 = \frac{2kp^2 k_m}{(2fk_d k_t{}^3)^{0.5}} = A_5 \exp(-E_5/RT) \qquad (5.9)$$

$$K_6 = \frac{(2 + \nu)\ kp^2}{k_t} = A_6 \exp(-E_6/RT) \qquad (5.10)$$

$$Cf = 2\ f\ (1 - \nu/2) \qquad (5.11)$$

$$\nu = k_t c\ /\ k_t \qquad (5.12)$$

where [I] and [M] are concentrations of initiator and monomer respectively , x is the monomer conversion, t is the reaction time, A_i's are frequency factors, E_i's are activation energies, R is the constant of the ideal gases and T is the reaction temperature. The parameters μ_0 and μ_2 denote the zeroth and second moments of the dead polymer, respectively. The system of differential equations (5.2) to (5.4) can be integrated to give the variation of [I], x, μ_0 and μ_2 during the course of polymerization. The values of the different parameters, necessary for the integration, have been taken from Ponnuswamy (1984) and are listed in Table 5.2.

Table 5.2 Numeric values of model parameters

Parameter	Value	Parameter	Value
A_d	3.751 E16	E_d	33740
Ap	1.201 E9	Ep	9630
A_t	2.113 E8	E_t	1014
A_m	1.071 E10	E_m	18394
A_s	7.85 E7	E_s	15000
ν	0.5	f	0.5

The parameters ν and f are dimensionless while the units of the A_i's and E_i's parameters are in and respectively.

In a polymerization process, someone is interested in the molecular weight of the final product. The interesting and useful properties which are uniquely associated with polymeric materials are a consequence of their molecular weight. In most instances, there is some molecular weight range for which a given polymer property will be optimum for a particular application. So, molecular weight control is of prime importance in a polymer process. Between the different types of molecular weights, the number average molecular weight and the weight average molecular weight are the most common ones. The number average molecular weight M_n is defined as the weight of all the molecules in a polymer sample divided by the total number of molecules present and is given by:

$$M_n = \sum_{x=1}^{x=\infty} N_x M_x \qquad (5.13)$$

where N_x and M_x are the mole fraction and molecular weight of molecules whose chain length is x, respectively. If we use the weight fraction w_x instead of the mole fraction N_x, in equation (5.13) the weight average molecular weight Mw and is given by:

$$Mw = \sum_{x=1}^{x=\infty} w_x \, M_x \qquad (5.14)$$

Equations (5.13) and (5.14) are not useful in practical applications because of the infinite summation. In terms of the zeroth and second moments of the dead polymer, the previous molecular weight averages are given as:

$$M_n = \frac{MW \, [M_0] \, x}{\mu_0} \qquad (5.15)$$

and

$$Mw = \frac{MW \, \mu_2}{[M_0] \, x} \qquad (5.16)$$

where MW is the molecular weight of the monomer and $[M_0]$ is its initial concentration.

So far, we have described a mathematical model for the solution polymerization of methylmethacrylate. This model gives the variation of [I], x, μ_0, μ_2, M_n, Mw during the course of polymerization. The control of the conversion both experimentally and by simulation will be studied for the batch polymerization reactor of methylmethacrylate by applying different control strategies. Between the two molecular weight averages, the weight average molecular weight will be controlled by simulation. These simulation and experimental results will be presented in the next sections.

5.5 Simulation Results

Several simulation results of the methylmethacrylate polymerization system were made under pole-assignment (PA), adaptive PID based on pole-assignment, self-tuning (STC) and fixed gain PID control. The objective is to compare the performance of the different control algorithms by controlling the monomer conversion or the weight average molecular weight of the polymer produced within the reactor. The kinetic model described in the previous section was used to simulate the variation of the controlled variables with respect to the temperature which is used as the manipulated variable. In this simulation study, we do not take into account the dynamics of the reactor. We assume that the temperature defined by the control algorithm can be automatically achieved within the reactor by some means (hot or cold water). Of course, this is unrealistic but still we can get some conclusions about the performance of the different control algorithms. The kinetic constants used in the simulation studies are given in Table 5.2. The initial monomer concentration was 4.75 moles/lit while the corresponding solvent concentration was 4.7065 moles/lit. These concentrations correspond to a solvent to monomer volume fraction of 0.5. Because of heating or cooling limitations, the manipulated variable is allowed to vary between 328°K and 353°K. For almost all the simulation runs, we assume that the initial temperature of the reactants is 340°K. Details of the control runs are summarized in Tables 5.3 and 5.4.

In the first case (see Table 5.3) our objective is to follow a desired conversion trajectory. According to our objective, the monomer conversion must follow a linear trajectory. The control of conversion in a batch reactor system is important because in a minimal time optimal problem the solution is to follow a 'ramp-type' conversion trajectory. The rate at which the monomer conversion varies with respect to time must be consistent with the heat removal capacity of the system.

Table 5.3 Details of monomer conversion control runs

	Control algorithm type with parameters used			
Run type	Pole assignment algorithm (P4)	Adaptive PID	Self-tuning controller (STC)	Fixed gain PID
Deterministic case with 80% desired final conversion after 150 min	$\eta_a=2$, $\eta_b = 2$ $\underline{\theta}(0)=[0,0,0,0]$ $\underline{P}(0)=10^6\cdot\underline{I}$ $u_f=340°K$ pole at $z=0.5$	Same conditions as in the case of pole assignment except that pole at $z=0$ Figures 5.13 to 5.16	$\eta_a=2$, $\eta_b=2$ $\underline{\theta}(0)=[0,0,0,0]$ $\zeta = 0.01$ $\lambda = 0.95$ $\tau_s = 30.5$ Figures 5.22 to 5.25	$K_p = 90$ $\tau_i = 20.5$ $\tau_d = 60.5$ $\tau_s = 30.5$ Figures 5.31, 5.32
Stochastic case (corresponds to addition of zero mean 0.005 standard deviation measurement noise to the measured conversion	Same conditions as above Figure 5.8	Same conditions as above Figure 5.19	Same conditions as above Figure 5.28	Same conditions as above Figure 5.35
Deterministic case with initiator addition at time t=48.5 min	Same conditions as in the deterministic case Figures 5.9, 5.10	Same conditions as in the deterministic case Figures 5.17, 5.18	Same conditions as in the deterministic case Figures 5.26, 5.27	Same conditions as in the deterministic case Figures 5.33, 5.34
Deterministic case with one sudden set point change at time t=120 min and 75% desired final monomer conversion	Same conditions as in the deterministic case Figure 5.11	Same conditions as in the deterministic case Figure 5.20	Same conditions as in the deterministic case Figure 5.29	Same conditions as in the deterministic case Figure 5.36
Deterministic case with two sudden set point changes at times t=15 min and t=120 min and 75% desired final monomer conversion	Same conditions as in the deterministic case Figure 5.12	Same conditions as in the deterministic case Figure 5.21	Same conditions as in the deterministic case Figure 5.30	Same conditions as in the deterministic case Figure 5.37

The performance of every control algorithm, for the case of conversion control, is examined under different cases. We have examined different cases which are: the deterministic case, the stochastic case, the deterministic case with sudden initiator addition and the deterministic case with one or two set point changes.

Figures 5.3 to 5.12 are referred to the pole-assignment control algorithm. We assume that the relation between conversion and temperature can be adequately described by a second order ARMA model. This model has the form

$$y(k)=a_1y(k-1)+a_2y(k-2)+b_1u(k-1)+b_2u(k-2)$$

where the time delay is zero. The process output $y(k)$ represents the monomer conversion at the discrete time k while $u(k)$ stands for temperature. Since there is a significant difference between the values that the conversion and the temperature take on, the regressor will contain very small and very large values. This created problems for the estimation of the process parameters. Because of it, an incremental model of the following form

$$y_i(k)=a_1y_i(k-1)+a_2y_i(k-2)+b_1u_i(k-1)+b_2u_i(k-2)$$

is used. The incremental variables $y_i(k)$ and $u_i(k)$ are defined as

$$y_i(k)=y(k)-y(k-1)$$

and

$$u_i(k)=u(k)-u(k-1)$$

Figure 5.3 shows the performance of the pole assignment control algorithm for the deterministic case. We can see that the controller is able to exactly follow the desired monomer conversion trajectory up to 135 minutes. For times greater than 135 minutes, the controller cannot follow the desired conversion trajectory. Despite the fact that the temperature is at its upper limit, the control objective cannot be achieved. This is due to the monomer depletion. During the initial phase of the reaction, we observe an increase in the temperature (process input) profile which is followed by a small decrease. This can be explained as follows: the temperature initially increases because the reactants must be brought to a temperature sufficient for the ignition of the reaction. After the reaction starts, the heat of the exothermic reaction causes a drop in the total required amount of energy necessary to sustain the desired reaction rate. This drop in the energy demand is translated into a temperature decrease. The otherwise increasing temperature profile is due to the monomer consumption. By assuming that the initial temperature of the reactants is 340°K, which is close to the ignition temperature, we significantly shorten the initial heating phase. Because of it we do not observe any delay in the increase of the conversion.

Figure 5.4 corresponds to the conditions of figure 5.3 and shows the variation of the initiator concentration with respect to time. According to equation (5.1) the initiator concentration should follow an exponential trajectory. The profile in figure 5.4 is due to the increasing values of the dissociation constant k_d because of the increasing temperature profile.

Figures 5.5 and 5.6 shows the process and controller parameters. The divergence of the parameters during the last time instants is due to the monomer depletion. Figure 5.7 compared to figure 5.3 emphasizes the fact that the performance of the pole-assignment algorithm does not heavily depend on the pole location.

In figure 5.8 we examine the performance of the controller in a stochastic environment. A Gaussian noise with zero mean and deviation of 0.005 is added to the measured conversion. From the previously mentioned figure, we conclude that the controller can achieve its objective but in the expense of an oscillatory input trajectory. Figure 5.9 compared to figure 5.3 illustrates the effect of the initiator on the performance of the controller and on the whole process in general. As we previously mentioned, the initially used initiator concentration is equal to 0.05 moles/lit. During the course of polymerization the drop of the initiator concentration and the demand to keep a constant reaction rate creates an increasing temperature profile. At time t=48.5 min we suddenly increase the initiator concentration to 0.07 moles/lit (see figure 5.10). This automatically increases the concentration of the free radicals within the reactor. This means that the monomer consumption or in other words the polymerization rate would increase unless the temperature drops. Since the objective is to keep constant the polymerization rate, the controller achieves this objective by decreasing the temperature (figure 5.9).

From figure 5.3, it is very obvious that the controller cannot achieve the final objective of 80% conversion after 150 minutes even if the upper limit of the temperature is used. It seems reasonable to reduce the desired final conversion to a lower value, i.e. 75%. So, at time t=120 min we decrease the rate of polymerization such that a final conversion of 75% results. Figure 5.11 shows that the controller causes the monomer conversion to exactly follow the desired trajectory without using the upper temperature limit. Figure 5.12 is similar to figure 5.11 but we initially do not demand a high polymerization rate. It can be seen that the controller achieves the desired objective. The initially observed decrease in the temperature profile (figure 5.12) is due to the lower desired polymerization rate. It seems that the initial temperature of 340°K is sufficiently large for this desired polymerization rate.

In the sequel, we shall repeat the previously discussed simulation runs by using different control algorithms.

Figures 5.13 to 5.21 use the discrete PID control algorithm derived in the previous chapter. By examining these figures we can say that the performance of the controller is quite satisfactory although this controller has not been designed for such control problems (tracking problems). We have to mention that the performance of this controller heavily depends on the pole location. Therefore this is an important tuning parameter. By trial and error, it was found that the best pole location corresponds to the origin of the unit circle. The deterioration of the control performance in the stochastic case (figure 5.19) is due to the violation of the design assumption (assumption 4.28, chapter four) of this controller.

Figures 5.22 to 5.30 are refereed to the self-tuning controller (STC). Based on these figures we can say that, in all the cases, the controller achieves its objectives. Compared to the pole-assignment and the adaptive PID, its control action is more oscillatory. A value for the constraining factor ξ larger than 0.01 (figure 5.24) gives a smoother control action. Figure 5.25 shows the trajectory of the controller parameters. Because of the time varying nature of the process, the parameters do not converge but still the controller achieves the control objective. By comparing figure 5.28 with figure 5.8 we can see that the pole-assignment control algorithm behaves better than the self-tuning controller in the stochastic case for this particular control application.

Figures 5.31 to 5.37 are referred to the fixed gain three term PID controller. By examining these figures we can say that, in all the cases, the performance of the PID controller is quite satisfactory. By comparing the performance of this controller with the performance of the previously analysed adaptive control algorithms, we can say that a very well tuned PID controller shows a control performance comparable to that of an advanced adaptive control algorithm for this particular problem.

After examining the performance of the different control algorithms, by controlling the monomer conversion, we shall present a series of simulation results where we are interested in controlling the weight average molecular weight. Molecular weight is of prime importance for the polymerization industry since it determines most of the properties of the final product. It is very obvious that a satisfactory control performance of a control algorithm is necessary before it is applied in to an industrial process. Because of it, the four previously discussed control algorithms will be tested by controlling the weight average molecular weight. Details of these simulation runs are summarized in Table 5.4. For adaptive control purposes, it is assumed that a second order without time delay ARMA model describes the relationship between the weight average molecular weight and the control variable which is the reaction temperature. For the same reasons as in the case of conversion control, an incremental model of the process representation is used in all the adaptive control algorithms.

TABLE 5.4 Details of weight average molecular weight control runs

Control algorithm type with parameters used

Pole assignment algorithm (PA)	Adaptive PID controller	Self-tuning controller (STC)	Fixed gain PID
na=2,nb=2 $\underline{\theta}(0)'=[0,0,0,0]$ $\underline{P}(0)=10^6 \cdot \underline{I}$ pole at z=0.5 Uf=340°k $t_s=30.s$ $Mw_s=150000$	Same conditions as in the case of pole assignment algorithm except that: pole at z=0.99	na=2, nb=2 $\underline{\theta}(0)'[0,0,0,0]$ $\underline{P}(0)=10^6 \cdot \underline{I}$ $\zeta=0.01$ $\lambda=0.95$ $t_s=30.s$ $Mw_s=150000$	kp=-0.001 if $t \leq 50$ min kp=-0.01 if $t>50$ min $t_i=20.s$ $t_d=60.s$ $t_s=30.s$ $Mw_s=150000$
Figures 5.38 to 5.41	Figures 5.42 to 5.44	Figures 5.45 to 5.47	Figures 5.48 to 5.49

Figures 5.38 to 5.41 are referred to the pole assignment control algorithm. From figure 5.38 we can see that the desired weight average molecular weight of 150000 is achieved from the very beginning of the control run. Despite the highly nonlinear and time varying nature of the process (see figures 5.40 and 5.41) the controller achieves the control objective. In figure 5.38 we can see that after some oscillations during the initial phase of the polymerization, the temperature follows a decreasing profile. By trying to keep constant the weight average molecular weight we are mainly interested in keeping constant the ratio between the polymerization rate and the termination rate. During the course of polymerization the rate of termination rate decreases because of the increasing

reaction mixture viscosity. So, it is necessary to decrease the rate of polymerization in order to keep constant the molecular weight profile. The need for having a decreasing polymerization rate profile explains the decreasing temperature profile. Figure 5.39 shows the variation of the number average molecular weight during the course of this polymerization control run. We have to notice that the ratio Mw/M_n is almost kept constant close to the value of two. Since the ratio Mw/M_n (polydispersity) is greater than one, the termination reaction takes place by both the coupling and disproportionation reactions.

Figures 5.42 and 5.44 are referred to the adaptive PID controller based on pole-assignment techniques. For this particular controller we can say that its control performance heavily depends on the pole location. By compromising between a fast control response and small overshoots (figure 5.42), we found that the best pole location is at $z=0.99$. Figure 5.44 shows the variation of the gains of the adaptive PID controller.

Figures 5.48 to 5.49 are referred to the fixed gain PID controller. Figure 5.48 shows the performance of the controller which can be said to be quite satisfactory. Really, this performance has been received by using a variable proportional gain. The proportional gain varies from -0.001 to -0.01 for times greater than 50 minutes. By initially using a small proportional gain we manage to avoid excessive overshoots. After the desired set point has been reached, the increased proportional gain helps in making the system to respond fast to any control errors. This fact emphasizes the need for using adaptive control in the case of nonlinear and time varying processes.

In summary, we can say that advanced adaptive control can be used for the control of the monomer conversion or the weight average molecular weight during the course of a polymerization reaction. The excellent performance of the analysed control algorithms, in this ideal environment, encourages us to apply them to an experimental pilot-plant batch polymerization reactor of methylmethacrylate.

5.6 Experimental Results

In the previous section we examined the performance of different control algorithms by controlling the monomer conversion and the weight average molecular weight during the polymerization of methylmethacrylate by simulation studies. Despite the ideal environment under which we tested the different controllers, these simulation studies gave an understanding about the polymerization process and a good promise for an at least satisfactory performance of the controllers in a real environment.

The experimental set-up used in this work has been described in the previous sections and can be seen in figure 5.1. For control purposes a cascaded control loop is used as in figure 5.2. The controller under consideration is used as the master controller. It defines a remote set point for a fixed gain PID controller used as the slave one. This slave controller is responsible by manipulating the flowrate of cold water to keep the reactor temperature close to that one defined by the master controller.

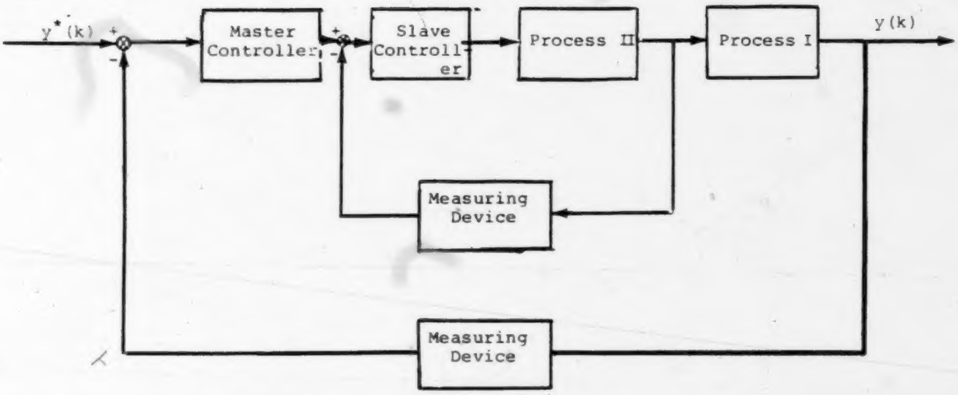

y*(k)
+
-
Master Controller
+
-
Slave Controller
Process II
Process I
y(k)
Measuring Device
Measuring Device

Figure 5.2 - Schematic Diagram of a Cascade Control Configuration

In our case, as Process I can be considered the monomer conversion while as Process II can be considered the reactor temperature. In the experimental studies three different controllers have been considered as master controllers: the pole-assignment algorithm (PA), the self-tuning controller (STC) and the fixed gain PID controller. Details of the control runs as summarized in Table 5.5

TABLE 5.5 Details of monomer conversion experimental control runs.

Control algorithm type with parameters used		
Pole assignment algorithm (PA)	Self-tuning controller (STC)	Fixed gain PID
na=3, nb=2 $\underline{\theta}(0)^t=$ $[0.96499, 0.275, -0.2576, 0.00282, 0.0001]$ $\underline{P}(0)=10^6\cdot\underline{I}$ Uf=341°K pole at z=-0.5, $t_s=30.s$ xf=80%, tf=150min Figures 5.60-5.61	na=3, nb=2 $\underline{\theta}(0)^t=$ $[1.26438, 0.24078, -0.53809, 0.01305, 0.00042]$ $\underline{P}(0)=10^6\cdot\underline{I}$, $\zeta=0.01$ $\lambda=0.95$, $T_s=30.s$ xf=80%, tf=150min Figures 5.54-5.53	kp=90 $t_i=20.s$ $t_d=60.s$ $t_s=30.s$ $t_s=30.s$ xf=80% tf=150min Figures 5.50-5.51
Same conditions as above except that $\underline{\theta}(0)^t=[0.68211, 0.30948, \overline{0}.0343, 0.0001]$ Incremental temperature limits were used Figures 5.62-5.64	na=3, nb=2 $\underline{\theta}(0)^t=$ $\overline{[}0., 0., 0., 0., 0.,]$ $\underline{P}(0)=10^6\cdot\underline{I}$, $\zeta=0.01$ xf=80%, tf=150 min Figures 5.54-5.56	
Same as above except that $\underline{\theta}(0)^t=$ $\overline{[}0., 0., 0., 0., 0.]$ PID was used up to 30.5 min Figure 5.65	Same as above except that na=2, nb=2 Figures 5.57-5.59	
Same as above except that na=3, nb=2, $\underline{\theta}(0)^t=[0., 0., 0., 0., 0.]$ PID was used up to 35 min Figure 5.66		

For each control run preweighted quantities of monomer (927 gr) and solvent (1294 gr) are loaded to the reactor. These quantities of reactants correspond to a solvent volume fraction which equals to 0.6. A large solvent volume fraction is used for two reasons: to facilitate the heat exchange and to eliminate the gel effect. Having loaded the reactants, the temperature within the reactor is raised to 333°K. At that time instant, a preweighted amount of catalyst (benzoylperoxide) equal to 32 gr is added to the reactants while at the same time instant control of the reaction is taken over. It must also be mentioned that the amount of catalyst used corresponds to a concentration which is equal to 0.05 moles/lit.

In the sequel, the different control runs will be discussed in the order that they have been performed.

Figures 5.50 and 5.51 correspond to monomer conversion control under PID control. Details of this control run are summarized in Table 5.5. The desired final monomer conversion was 80% while the final reaction time was 150 min. The desired conversion trajectory is a straight line (figure 5.50) corresponding to a constant reaction rate throughout the polymerization course. The same figure also shows the actual monomer conversion trajectory. Initially there is a nonzero value for the conversion which is due to the conversion-density correlation

$$x = 1021.756\,\rho - 880.587$$

used in this work. Despite this initial discrepancy between the conversion given by the correlation and the actual one, this correlation seems to be very good as it can be seen from Table 5.6. Because of the monomer depletion, the controller cannot follow the desired trajectory for times greater than 130 min., although the reaction temperature is kept at the maximum possible limit (figure 5.51). The same figure shows that the heating dynamics of the reactor are very slow compared to the cooling ones. These slow heating dynamics are responsible for the oscillations of the monomer conversion around its desired trajectory.

Figures 5.52 to 5.59 are referred to monomer conversion control when a self-tuning controller is used as the master controller. Figures 5.52 to 5.53 are referred to the case when na=3 and nb=2 while a nonzero set of initial parameters (Table 5.5) has been used. Based on measurements taken during this control run, Table 5.6 has been constructed. The same comments mentioned for the case of the PID controller are also applicable to this case. One serious problem encountered in this case and in all adaptive control runs, in general, is related to the parameter estimation when the process input saturates at its upper or lower limit. Since an incremental process model is used, the regressor will contain zeros during long periods of process input saturation. In this case, because of lack of excitation, the calculated process parameters will be in error. We solve this problem by introducing some kind of excitation into the regressor. Because of it, a stepwise stationary incremental process input is used in the regressor when the process input saturates. This way of solving the above mentioned problem has been proven efficient for both the self-tuning controller and pole-assignment control algorithms used in this work.

TABLE 5.6 Conversion given by GC analysis and density correlation

Time (min)	Density (gr/cm^3)	Conversion given by GC (%)	Conversion Given by the correlation
5	0.8648	1.15	3.028
20	0.8672	4.75	5.480
35	0.8776	15.33	16.106
50	0.8884	26.75	27.141
65	0.8941	33.52	32.965
80	0.9040	42.86	43.080
95	0.9089	47.38	48.291
110	0.9189	57.63	58.407
125	0.9269	65.91	66.479
140	0.9338	71.35	73.529
150	0.9377	76.10	77.514

Figures 5.54 to 5.56 are also referred to the self-tuning controller. Details of this control run are in Table 5.5. Our objective, by performing this control run, is to examine the effect of the initial parameters set on the performance of the controller. We can see from figure 5.54 that the performance of the controller does not depend on our choice for the initial parameters set. Figure 5.56 shows the trajectory of the controller parameters. Despite the large variation of its parameters (highly time-varying process), the controller performs well if we take into account the problems associated with the reactor design.

Once again, the accuracy of the conversion-density correlation can be seen from Table 5.7 which is based on measurements taken during this control run. Tables 5.6 and 5.7 gives us confidence regarding the accuracy with which the monomer conversion within the reactor is measured. The observed disagreement between the conversion given by gas chromatography and the conversion-density correlation can be considered as acceptable.

Next we want to examine the effect of the order of the controller on its performance regarding the control of the monomer conversion. Because of it, the previous control run was repeated when the structure of the controller had na=2 and nb=2. Figures 5.57 to 5.59 show the performance of the controller. Despite the reactor design limitations, a second order self-tuning controller gives good control performance. For this particular application, this change in the order of the controller did not significantly improve or deteriorate the control performance.

Figures 5.60 to 5.66 are referred to the monomer conversion control when the master controller was the pole-assignment algorithm. Details of these control runs are summarized in Table 5.5. According to the stability analysis, if a "good" set of initial process parameters is available or a fixed gain controller able to stabilize the closed loop system is used until the parameters converge to a "good" set of ones and the singularity of the Sylvester matrix is not the limiting point of it. It was assumed that the process can be described by an ARMA model with na=3 and nb=2. As in the case of the self-tuning controller, the time delay was assumed to be equal to zero. Figures 5.60 to 5.61 show the performance of the pole-assignment algorithm when a nonzero set of initial process parameters has has been used.

TABLE 5.7 Conversion given by GC analysis and density correlation

Time (min)	Density (gr/cm³)	Conversion given by GC (%)	Conversion Given by the correlation
5	0.8644	1.33	2.619
20	0.8692	7.31	7.523
35	0.8806	19.90	19.171
50	0.8858	25.46	24.484
65	0.8906	35.79	34.906
80	0.9023	41.97	41.343
95	0.9117	50.51	50.948
110	0.9187	57.42	58.100
125	0.9262	65.92	65.763
140	0.9326	71.48	72.303
150	0.9362	77.24	75.981

Because of these nonzero parameters, the fixed gain controller, Uf=341°K was not used during the whole polymerization course. By comparing figure 5.60 with figures 5.50, 5.52 and 5.57 we observe an improvement in the control performance for the case of the pole-assignment control algorithm. During the course of this control run, the process input follows a rather "bang-bang" type of trajectory (figure 5.61). We can avoid this "bang-bang" type of control by using incremental input limits. To prove this, the previous control run has been repeated with a nonzero set of initial paramters (see Table 5.5) different than that of the previous control run. The process input was allowed to vary at most 3°K between successive control intervals. Figure 5.62 shows the performance of the controller for this particular run. The initial conversion of the reaction mixture used in this run has already been 5.2% since the same reactants had been used in another unsuccessful run. After the initial period, the control performance can be considered as satisfactory. Figure 5.63 shows that the "bang-bang" type of control has been avoided because of the incremental input limits used. Figure 5.64 shows the trajectory of the process parameters. We observe a slowly time varying parameter trajectory with the b's parameters close to zero (low "gain" system). Our attempt to repeat the same control run with zero initial parameters was unsuccessful. This can be attributed to the slow variations of the process parameters (figure 5.64).

As we previously mentioned, instead of having "good" initial process parameters to operate this control algorithm, we can use a fixed gain controller able to stabilize the system until the parameters converge to some "good" values. In this case the indentification scheme must run in a passive mode. In our case, the fixed gain PID controller previously discussed was used as the fixed gain controller for the first 36.5 min of the polymerization course. The model of the process was assumed to have na=3 and nb=2 while a zero initial set of parameters was used. After t=36.5 min the pole-assignment algorithm started being used. The control performance for this particular run can be seen in figure 5.65. Compared to figure 5.60 we observe a deterioration in the control performance. The same control run has been repeated for a second order process model. Compared to figure 5.65, we can say that the deterioration in the control performance can be attributed to an inadequate description of the process by a second order model.

In summary, we can say that the performance of the different control algorithms is almost the same for this particular process. Slightly better control performance was obtained when the pole-assignment algorithm with nonzero initial parameters was used. An improvement in the performance of allcontrollers would be obtained if the heating dynamics of the controlled system were faster. This suggests that a modification to the existent experimental set-up is necessary. The use of a split-range valve could be one such modification.

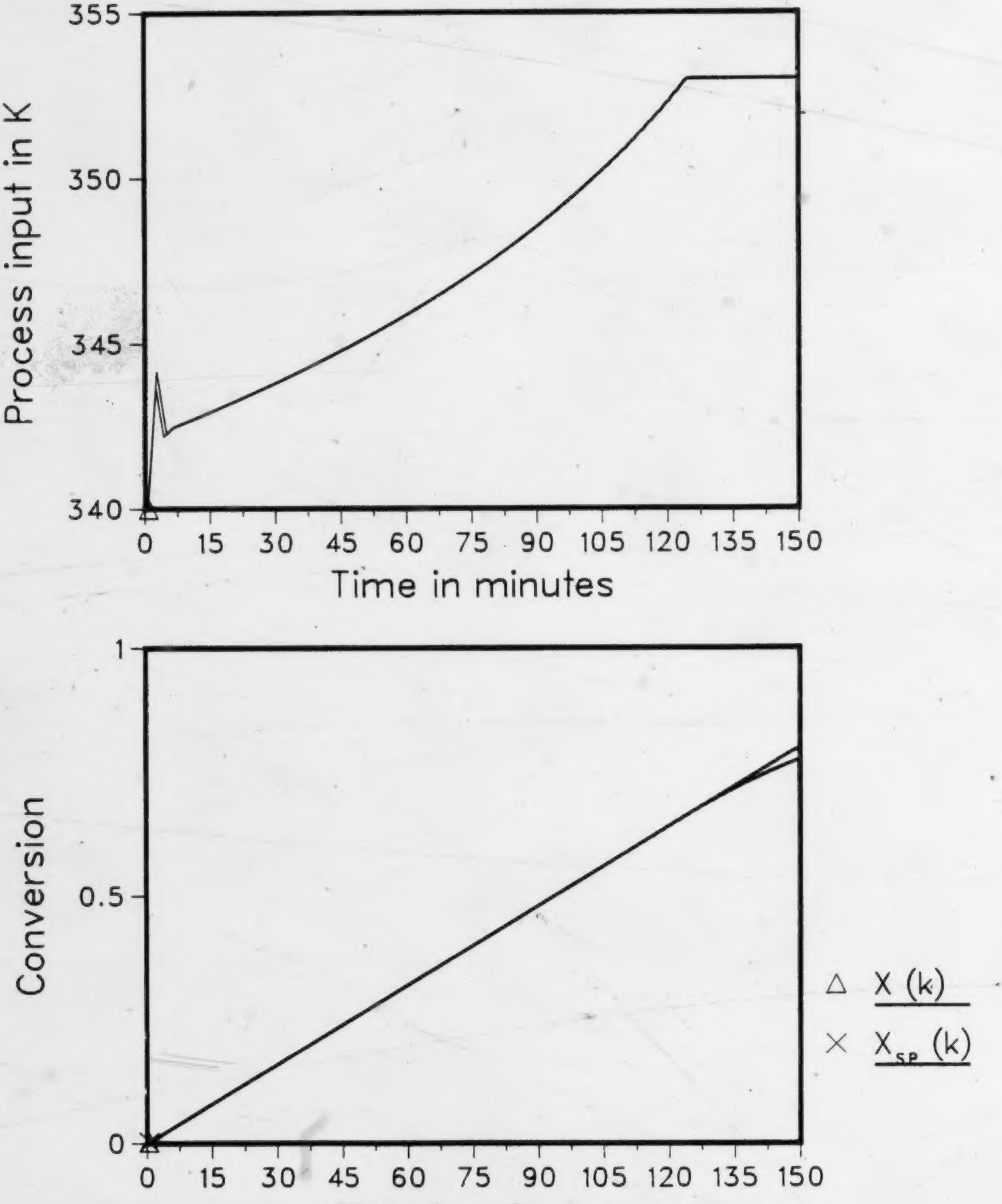


Fig.–5.3 Conversion control without set point changes PA/IA/DC/d=0/p=0.5/Uf=340.

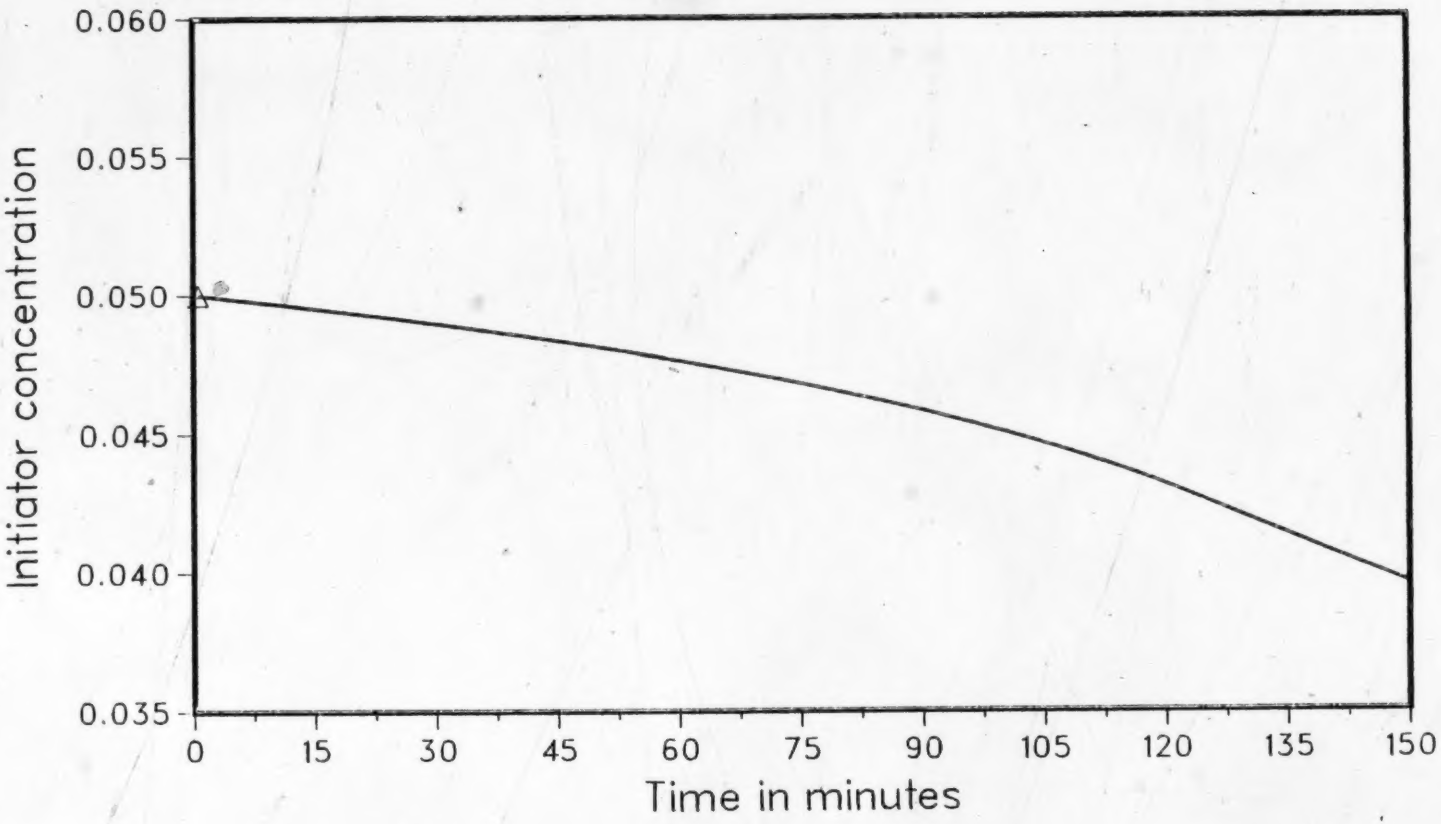


Fig. -5.4 Trajectory of initiator concentration
PA/IA/DC/d=0/p=0.5/Uf=340.

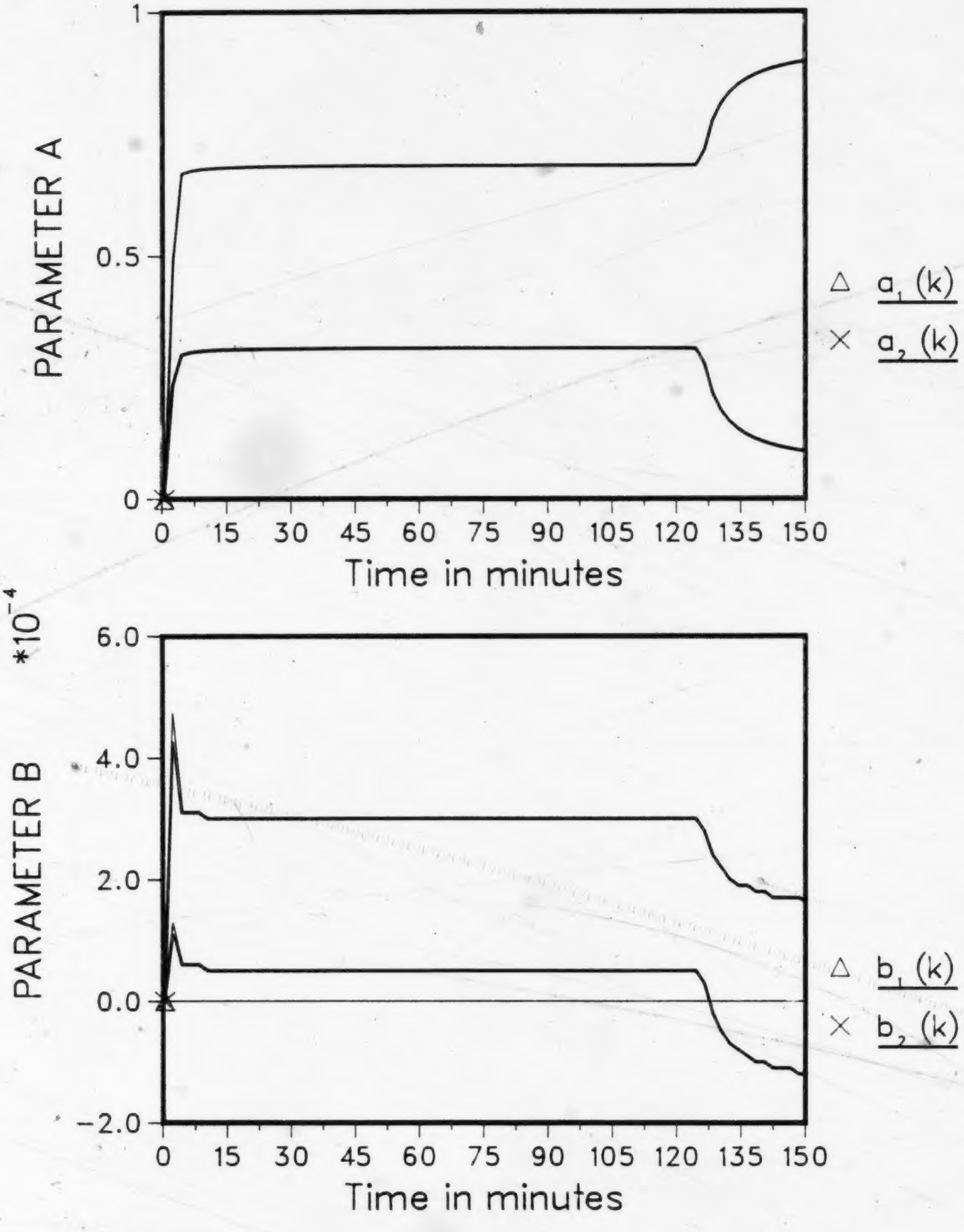


Fig.–5.5 Parameter trajectory
PA/IA/DC/d=0/p=0.5/Uf=340.

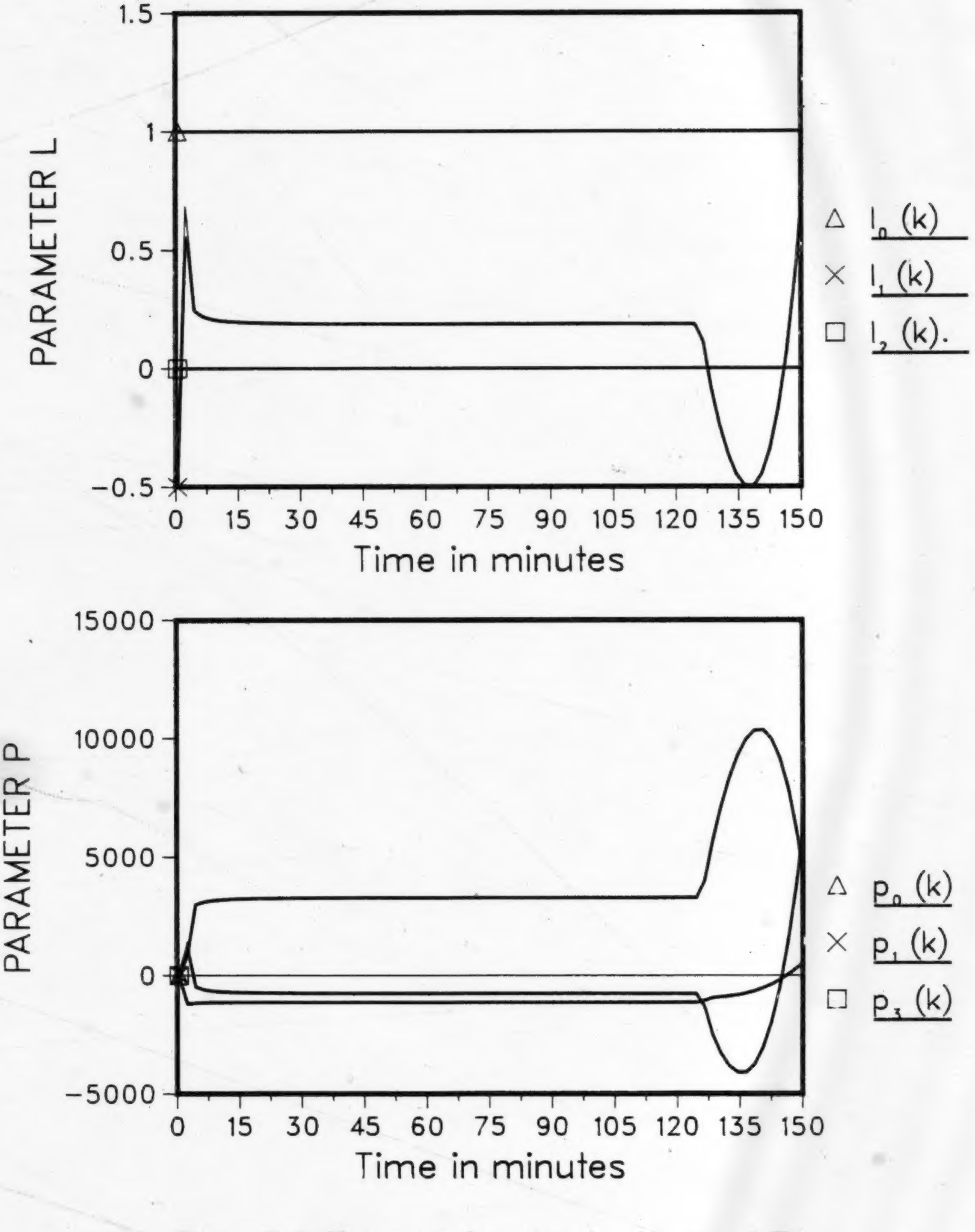


Fig.–5.6 Parameter trajectory of PA
PA/IA/DC/d=0/p=0.5/Uf=340.

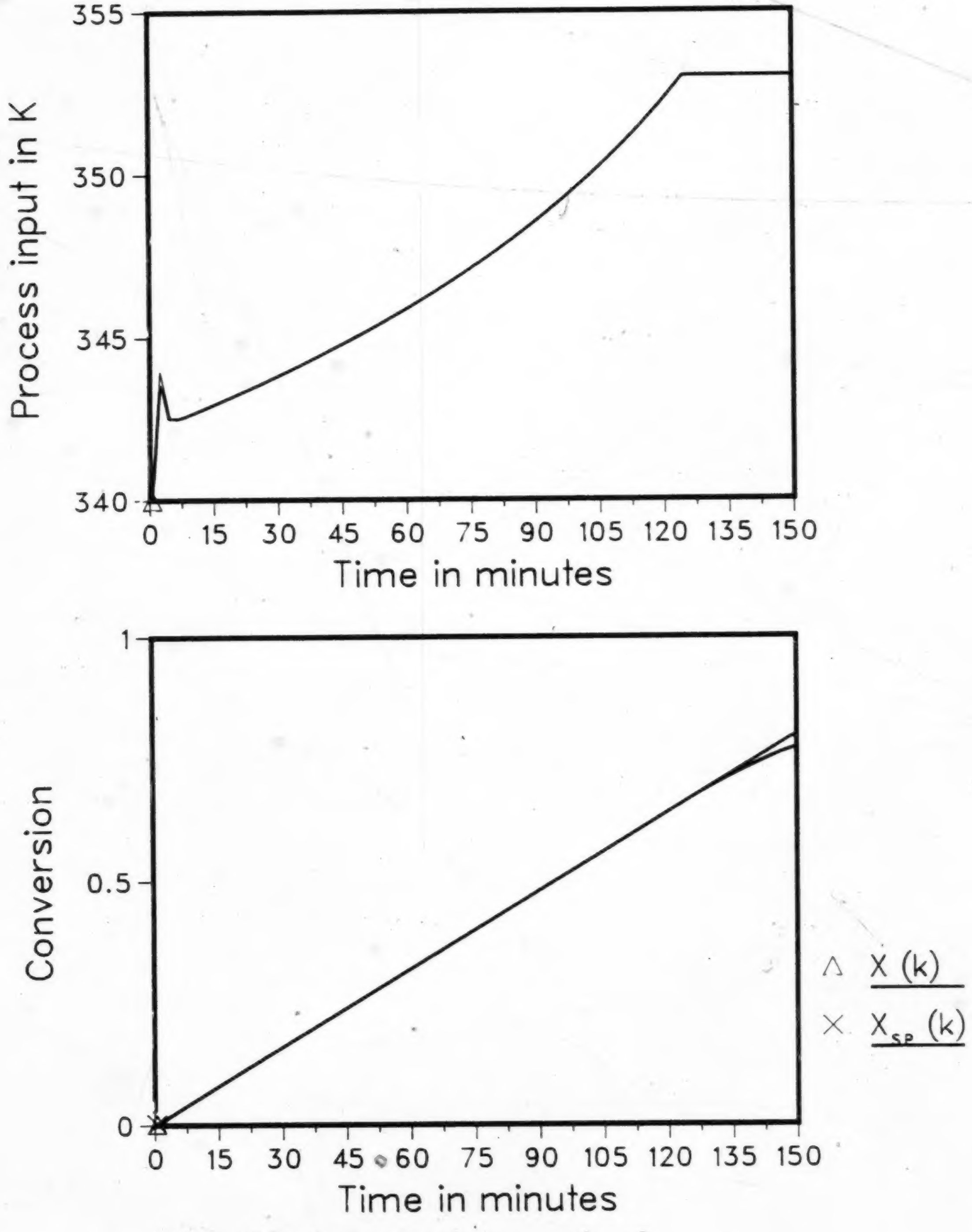


Fig.–5.7 Conversion control without set point changes PA/IA/DC/d=0/p=0./Uf=340.

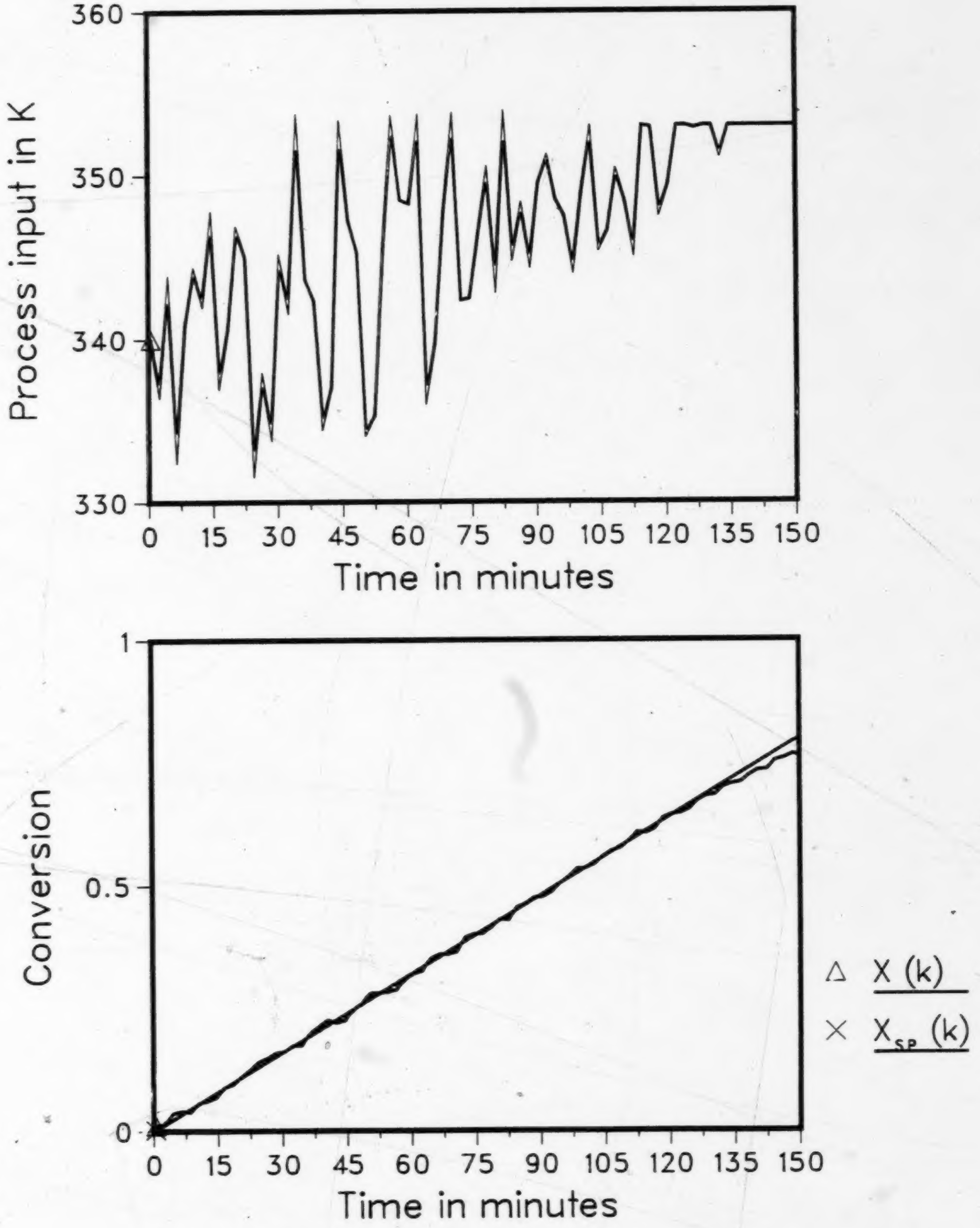


Fig.—5.8 Conversion control
without set point changes
PA/IA/SC.005/d=0/p=0.5/Uf=340.

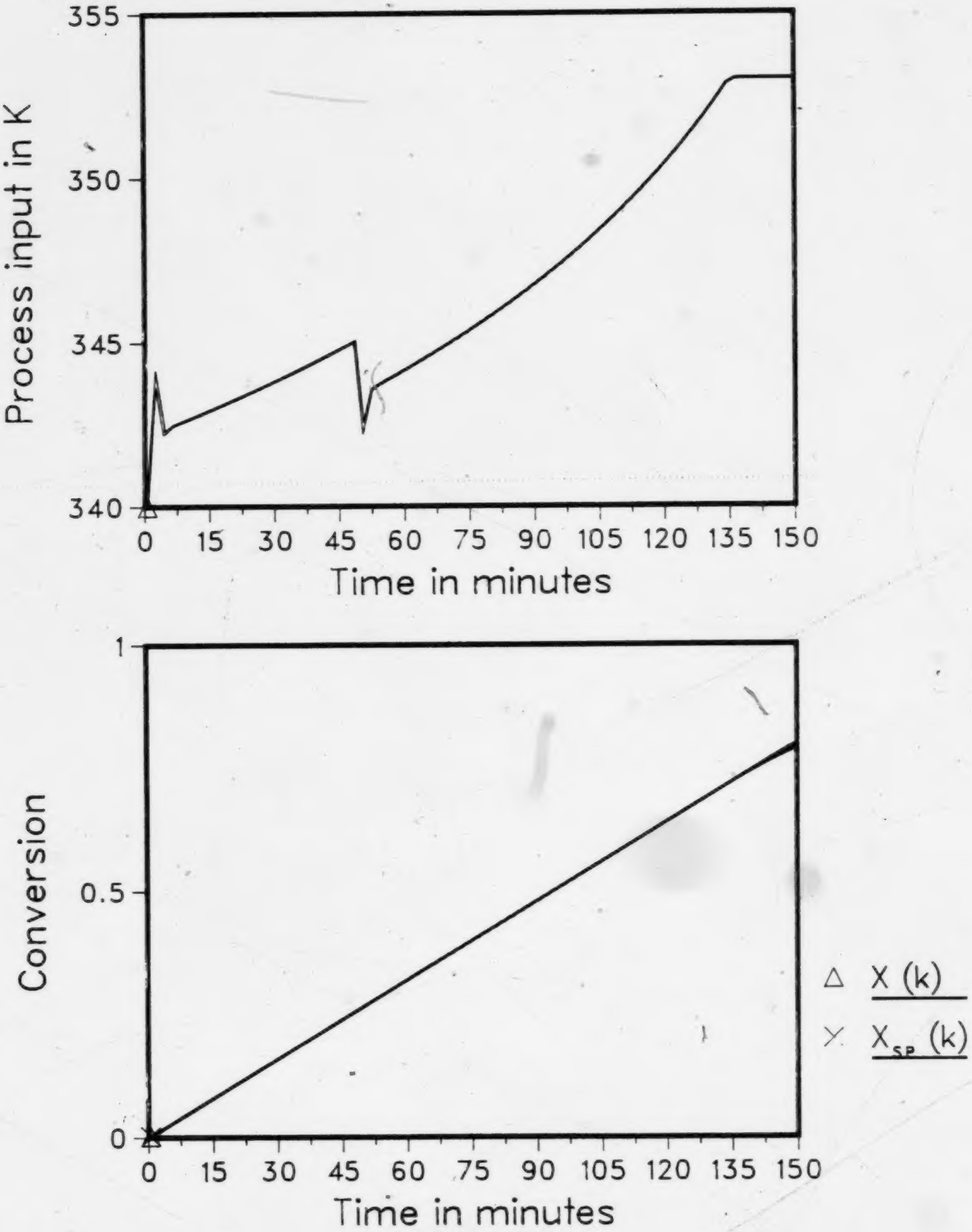


Fig.—5.9 Conversion control with initiator changes PA/IA/DC/d=0/p=0.5/Uf=340.

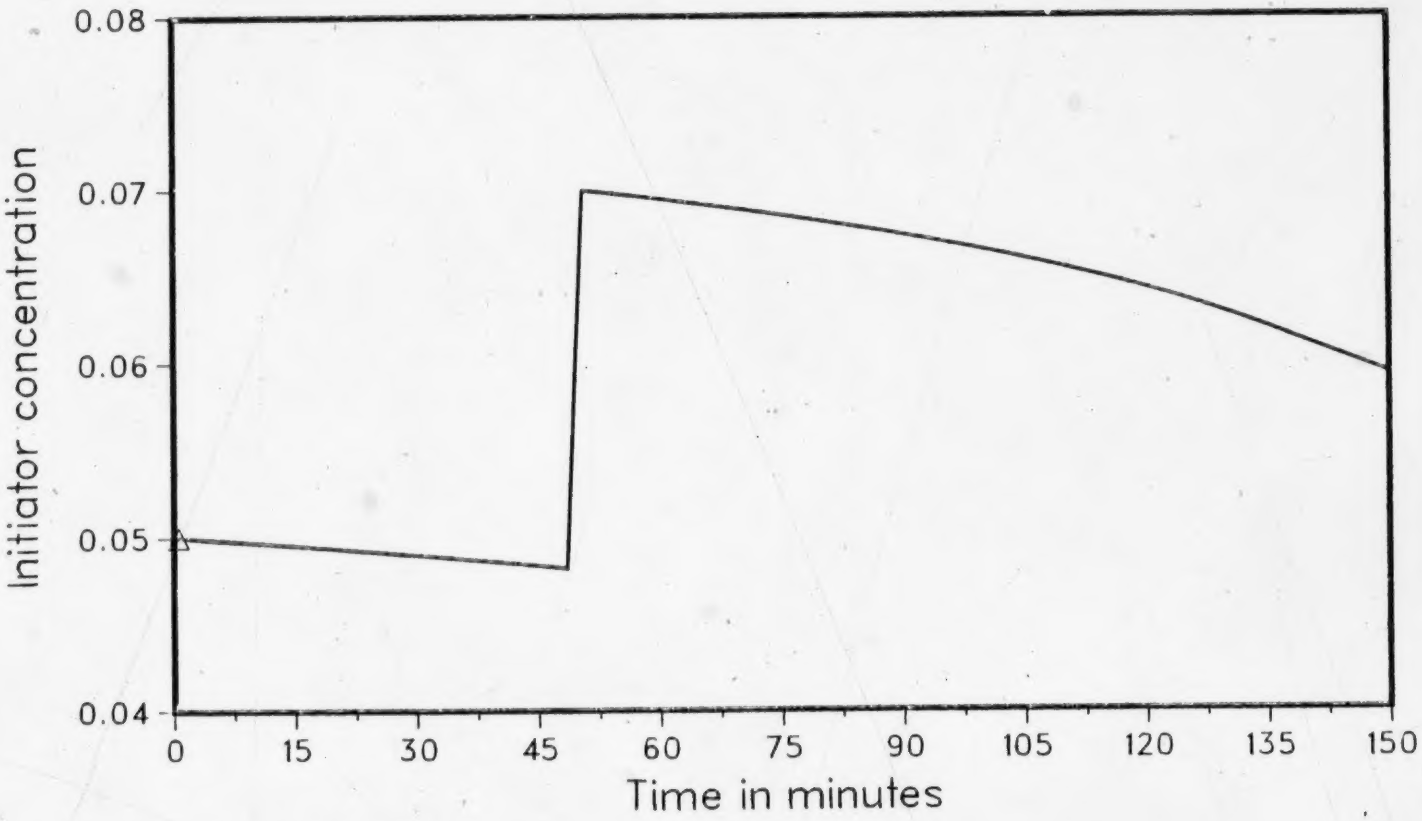


Fig. –5.10 Trajectory of initiator concentration with initiator addition at t=50 min PA/IA/DC/d=0/p=0.5/Uf=340.

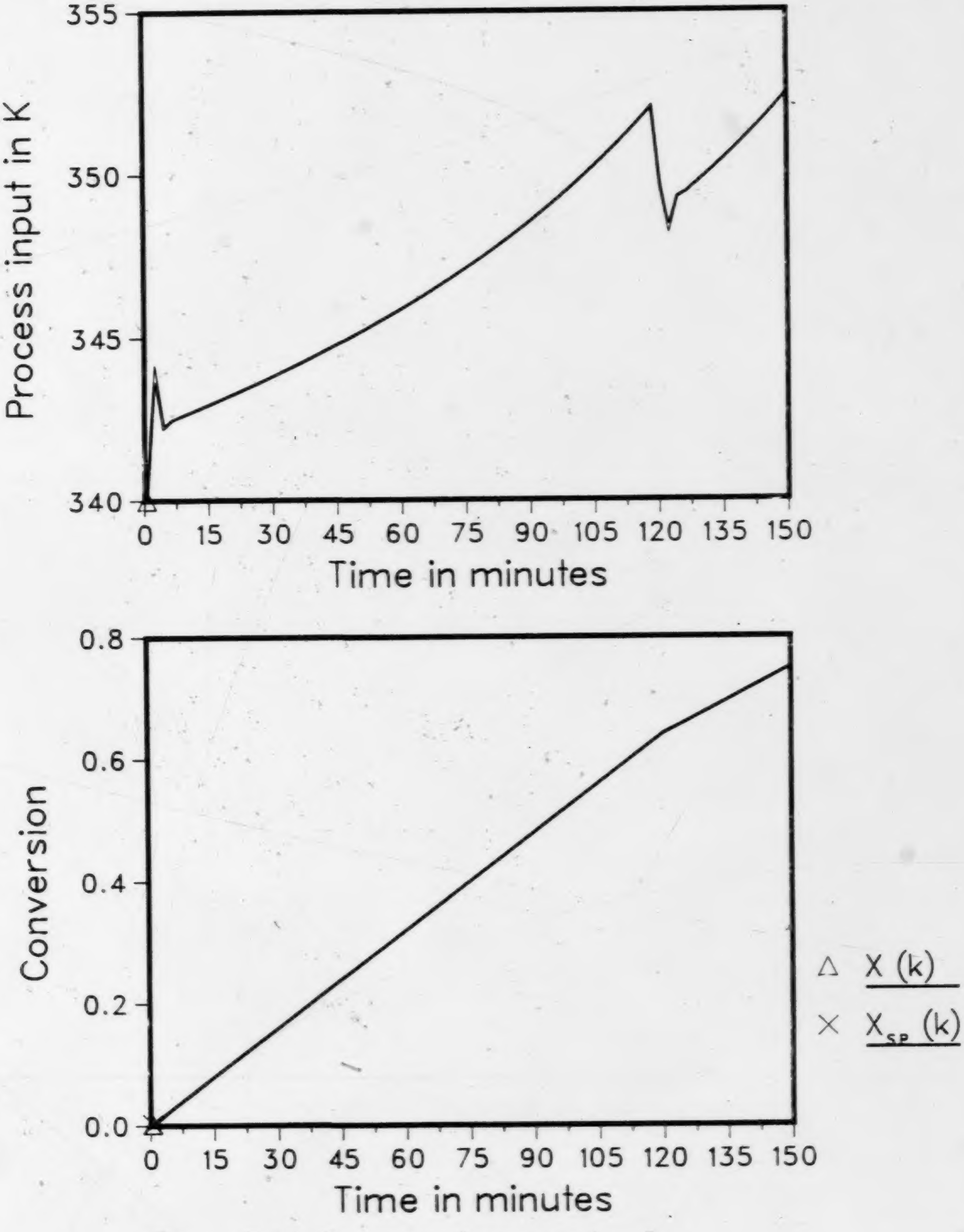


Fig.–5.11 Conversion control
with one set point change
PA/IA/DC/d=0/p=0.5/Uf=340.

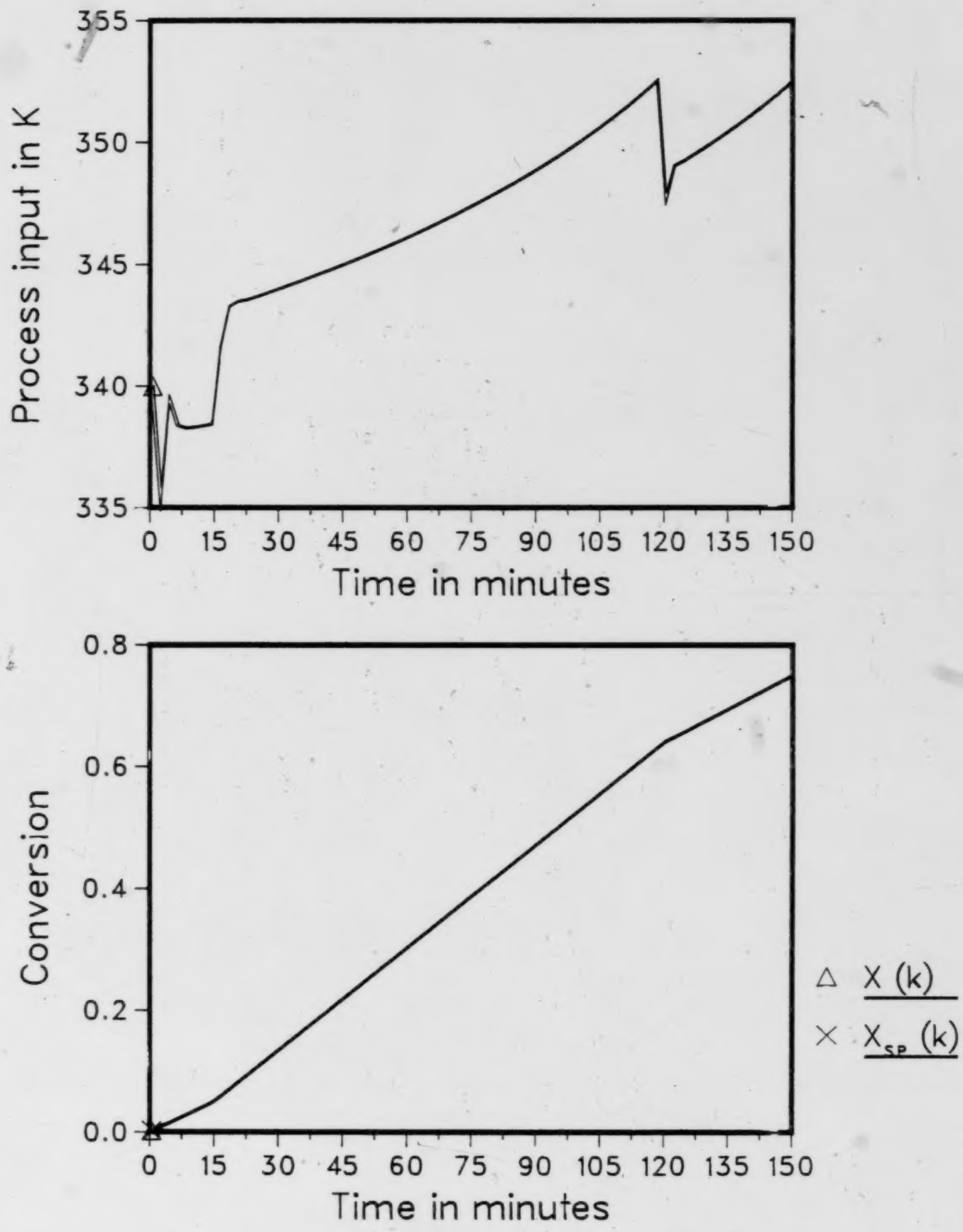


Fig.-5.12 Conversion control with two set point changes PA/IA/DC/d=0/p=0.5/Uf=340.

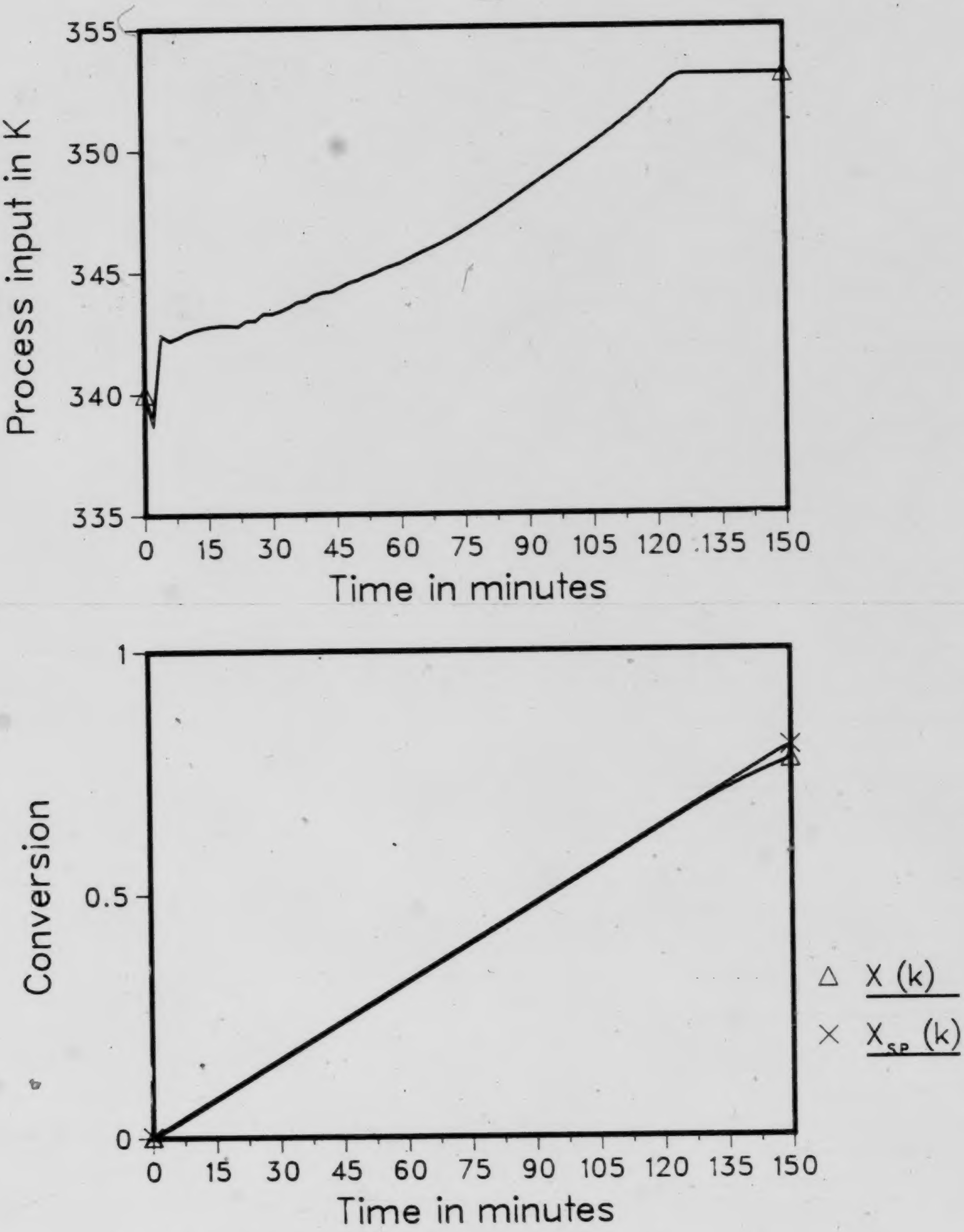


Fig.–5.13 Conversion control without set point changes PID/DC/d=0/p=0./Uf=340.

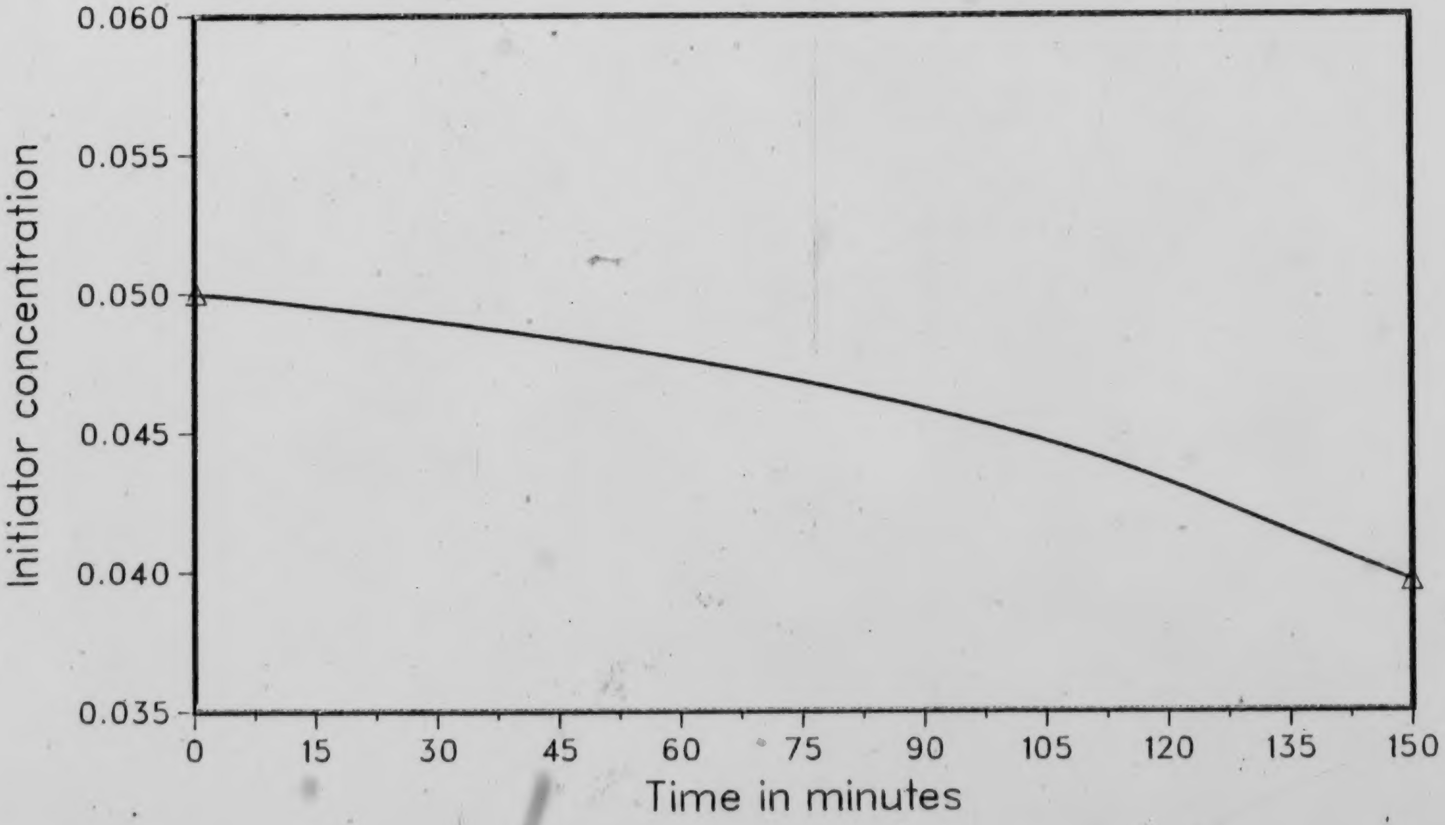


Fig. -5.14 Trajectory of initiator concentration
PID/DC/d=0/p=0./Uf=340.

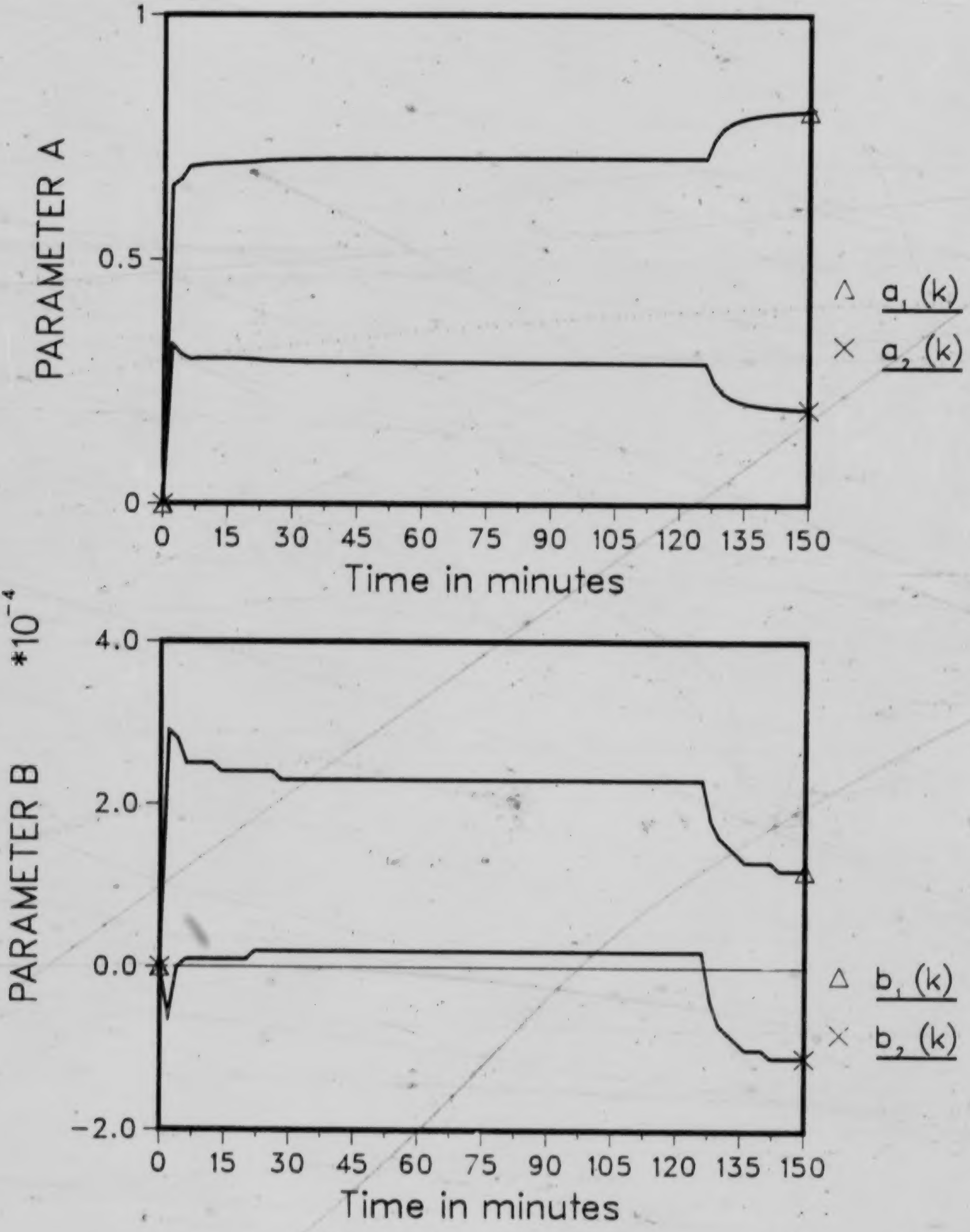


Fig.–5.15 Parameter trajectory
PID/DC/d=0/p=0./Uf=340.

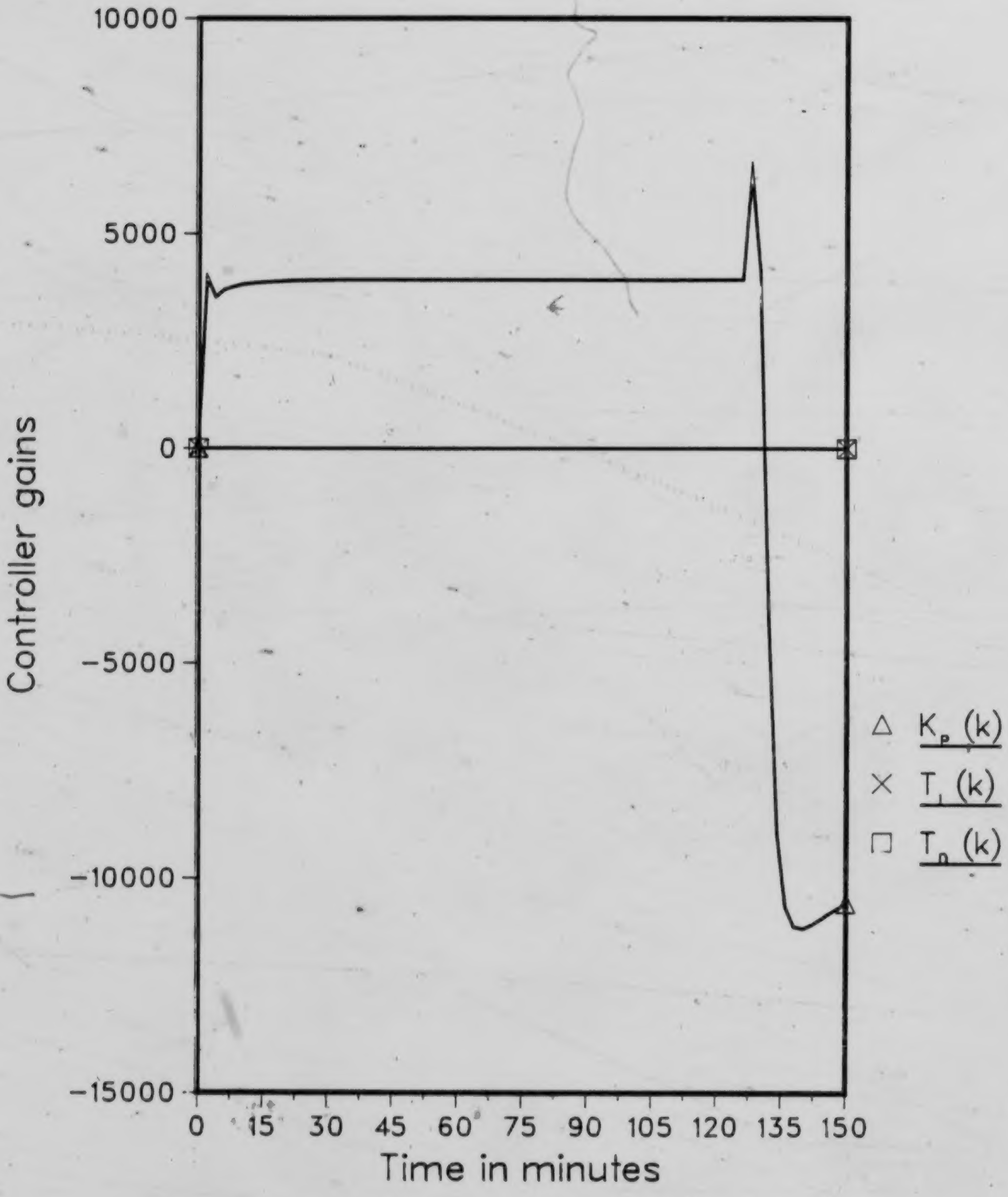


Fig.–5.16 Parameter trajectory of PID
PID/DC/d=0/p=0./Uf=340.

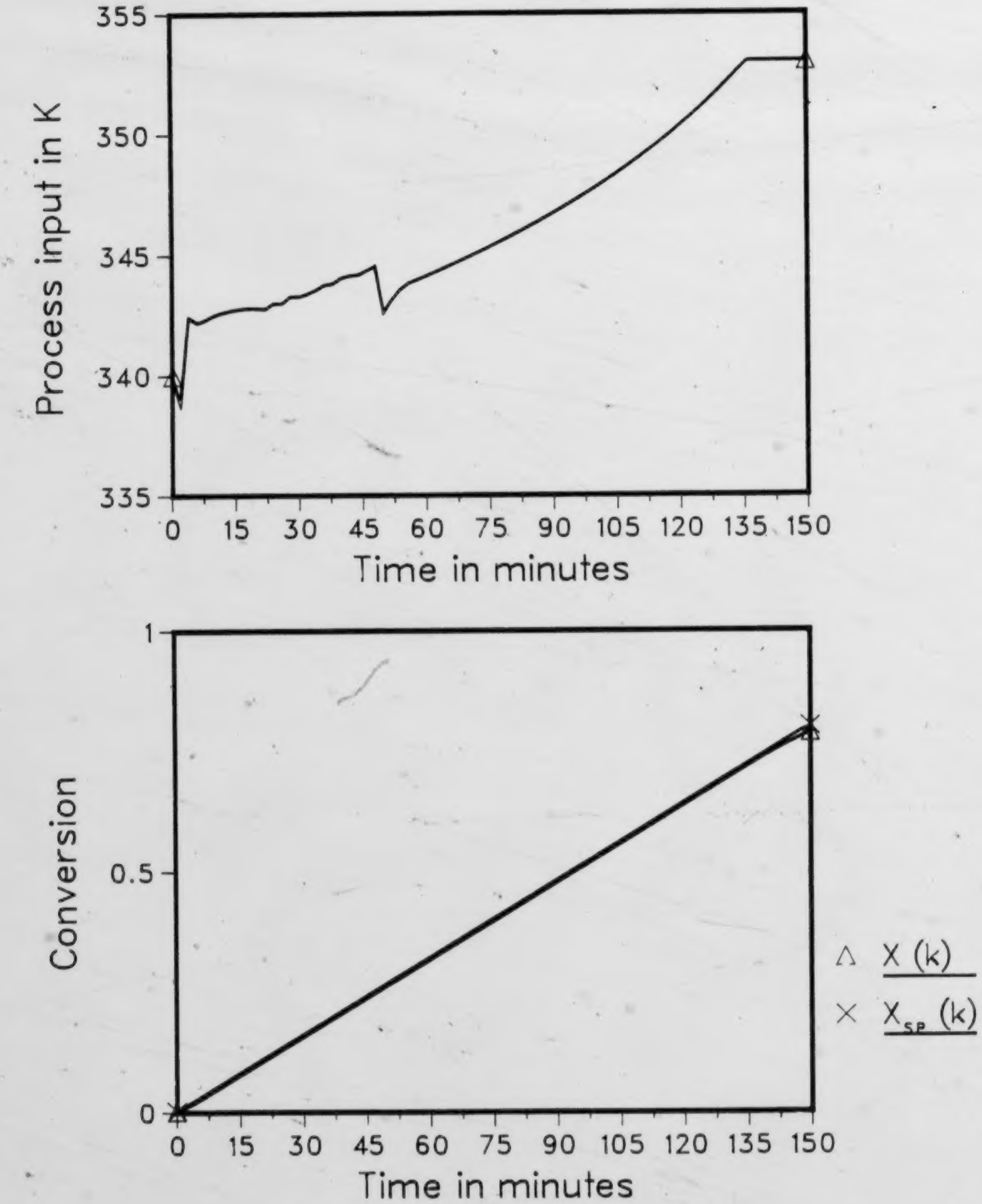


Fig.–5.17 Conversion control
with initiator changes
PID/DC/d=0/p=0./Uf=340.

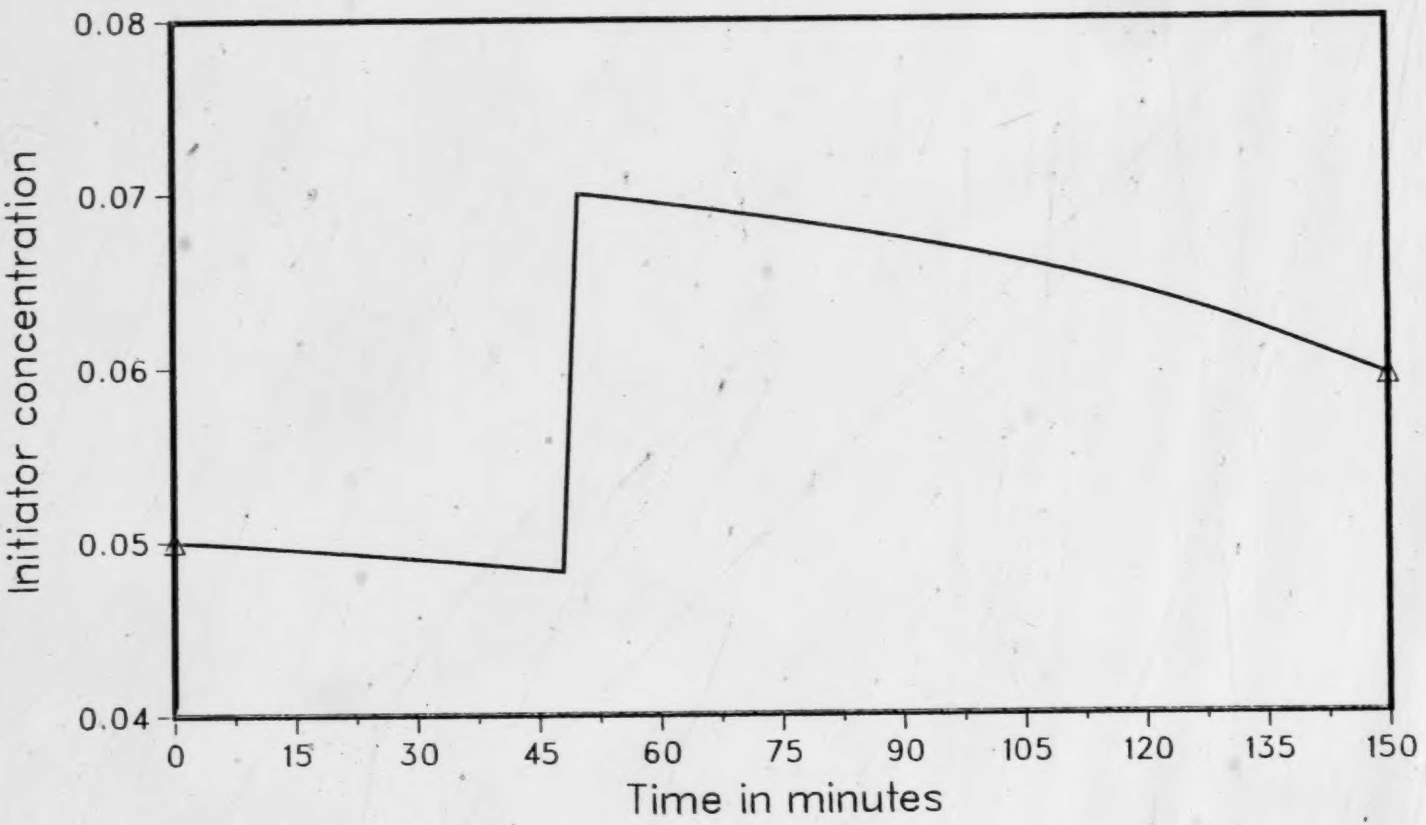


Fig. -5.18 Trajectory of initiator concentration with initiator addition at t=50. min. PID/DC/d=0/p=0./Uf=340.

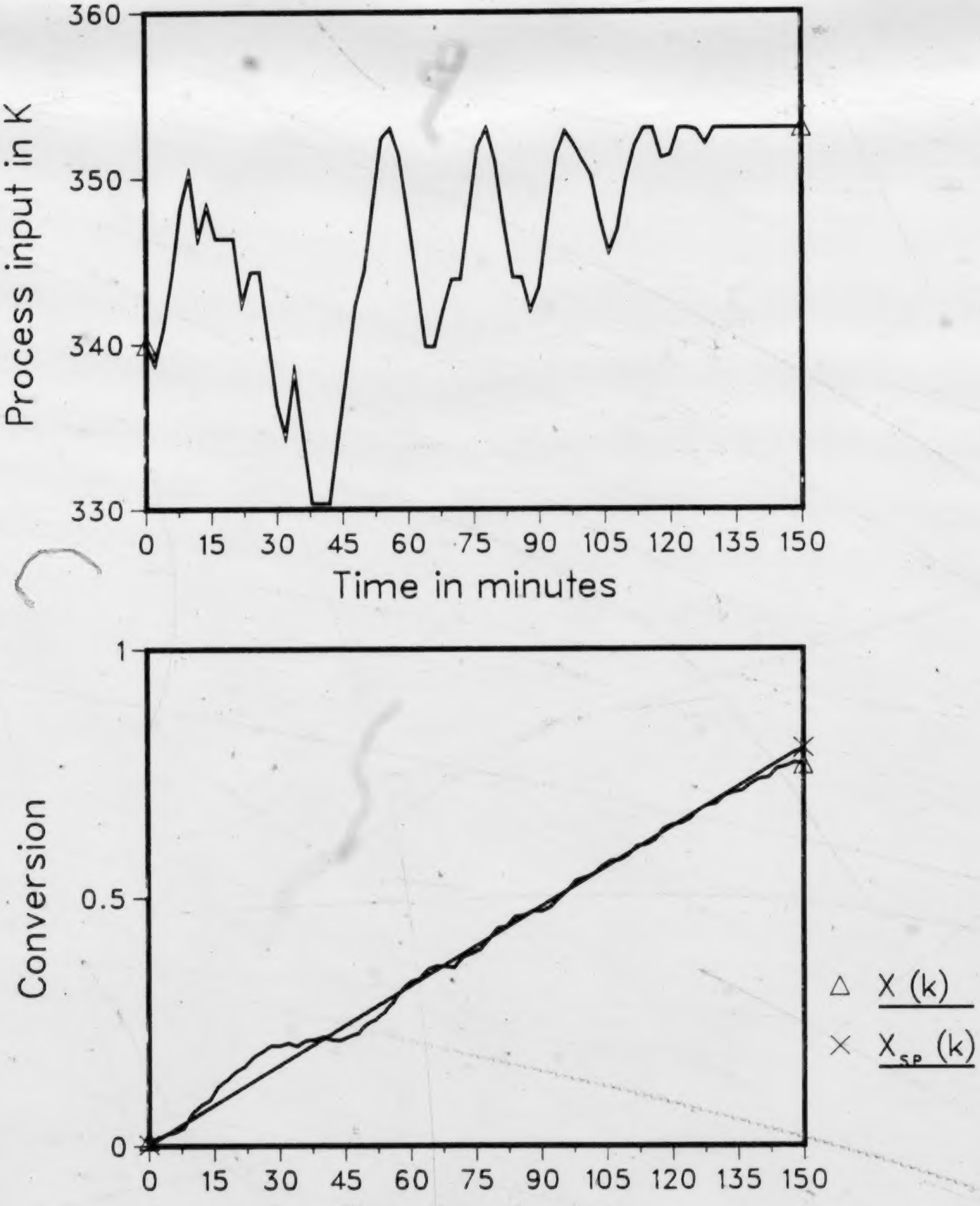


Fig.–5.19 Conversion control without set point changes PID/SC.005/d=0/p=0. /Uf=340.

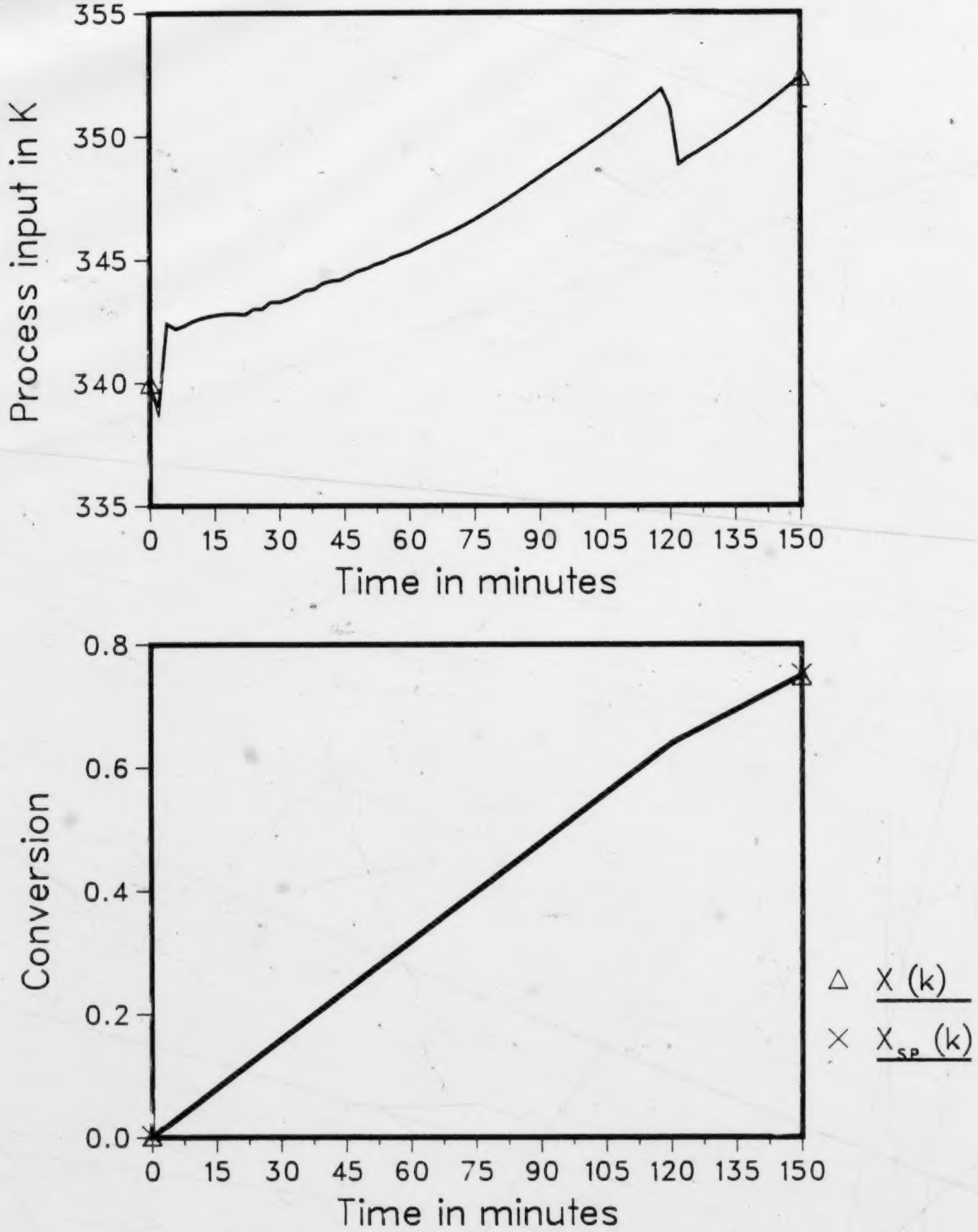


Fig.–5.20 Conversion control
with one set point change
PID/SC.005/d=0/p=0./Uf=340.

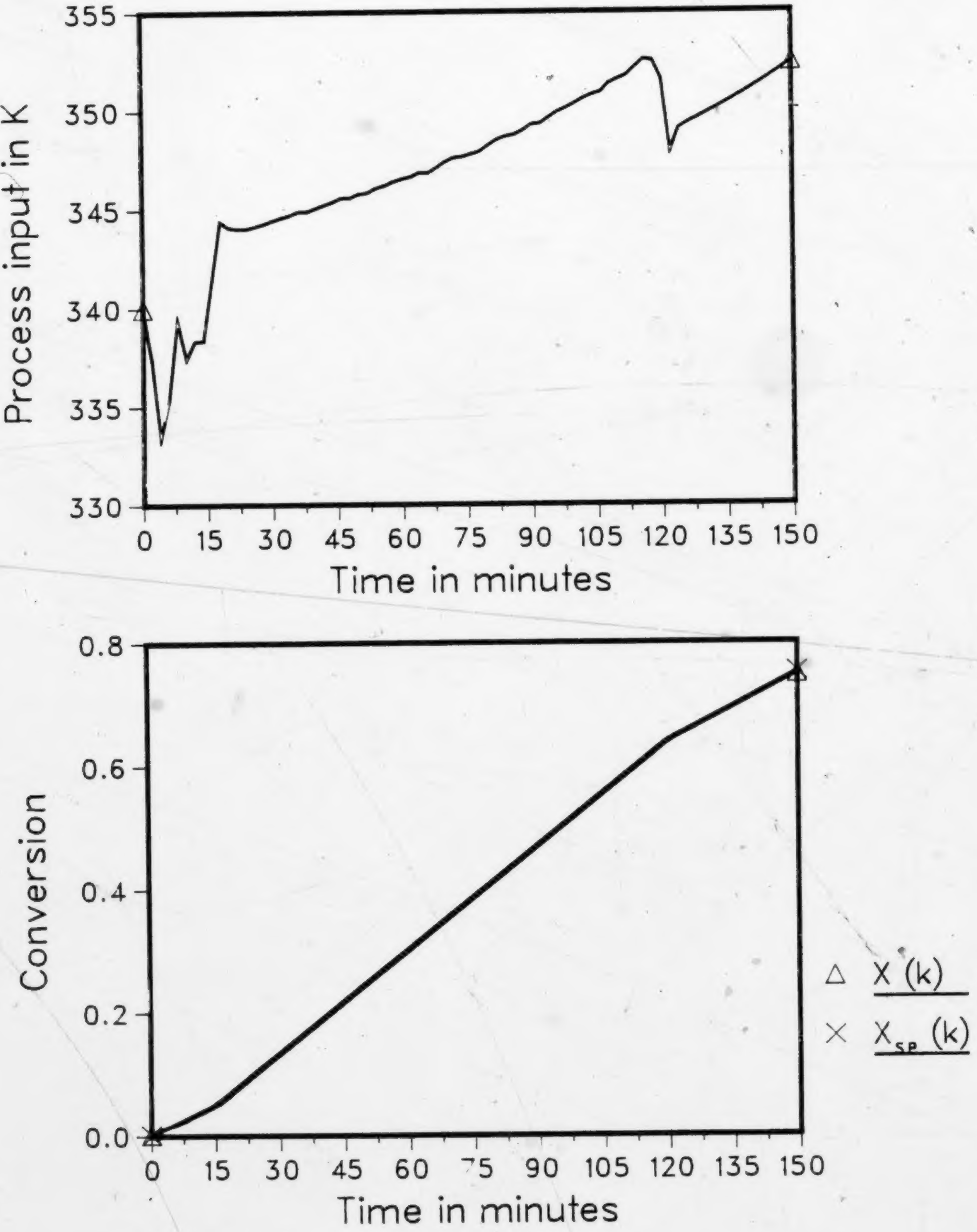


Fig.–5.21 Conversion control
with two set point changes
PID SC.005, d=0, p=0., Uf=340.

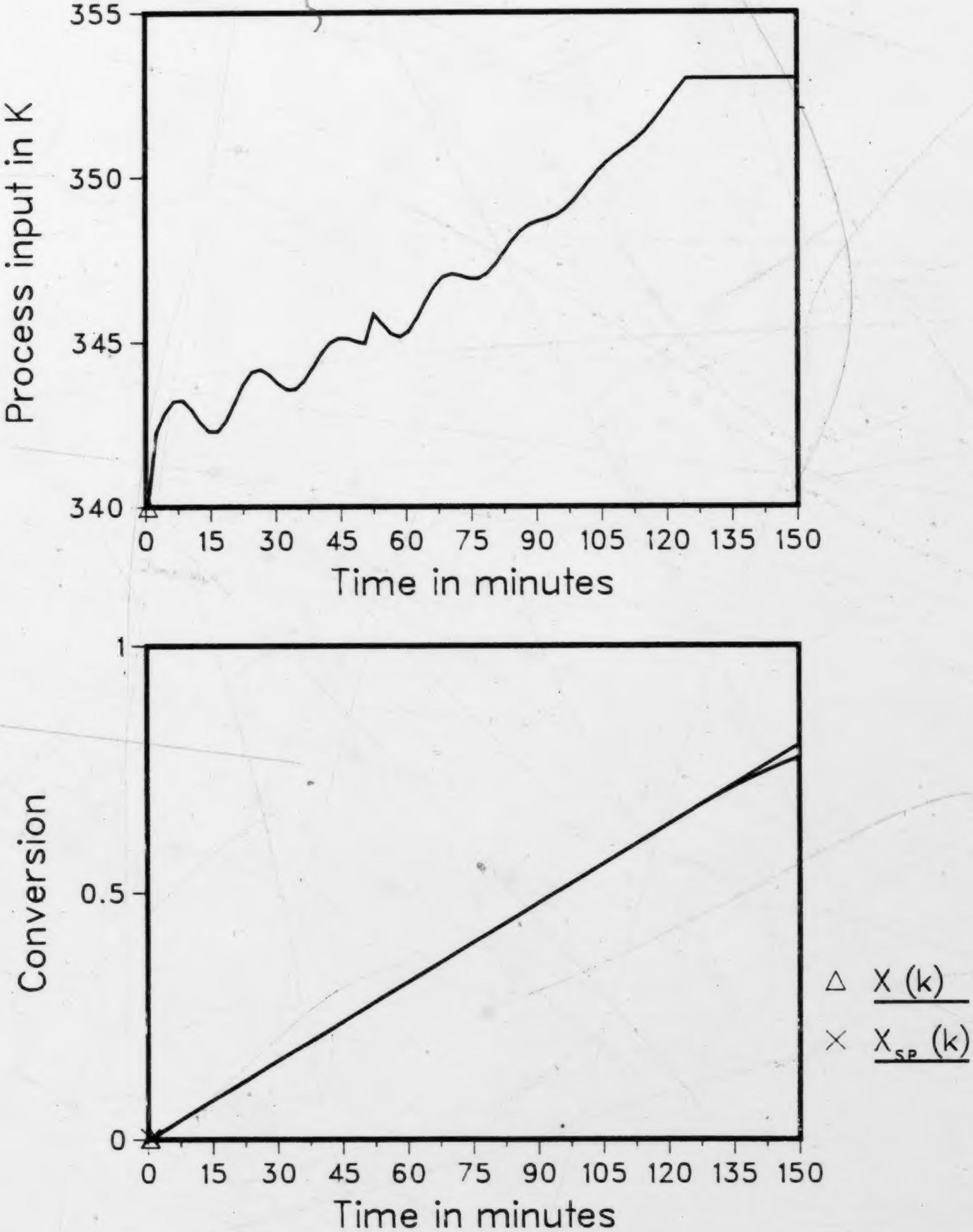


Fig.–5.22 Conversion control
without set point changes
STC/DC/d=0/

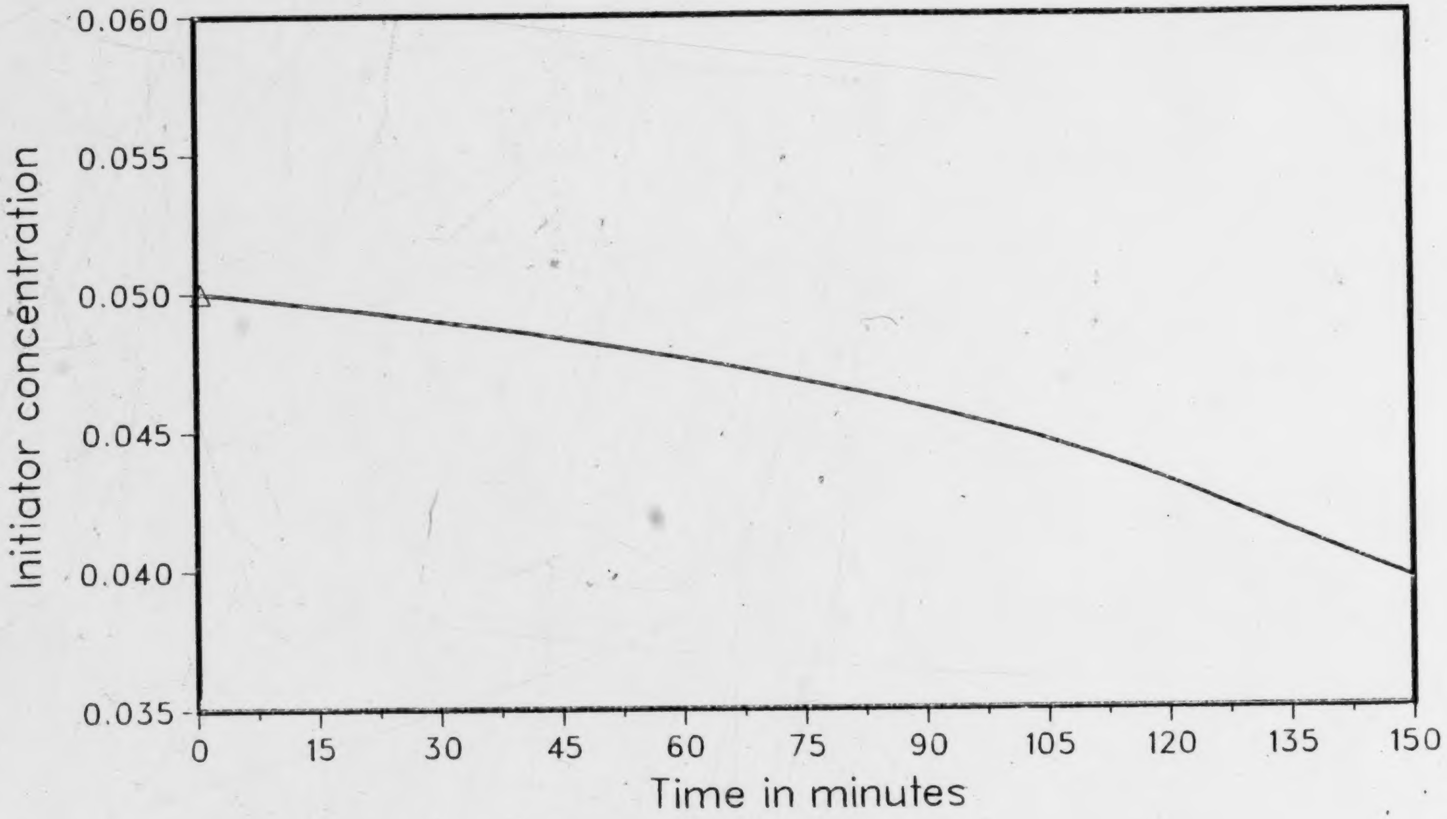


Fig. -5.23 Trajectory of initiator concentration with an STC

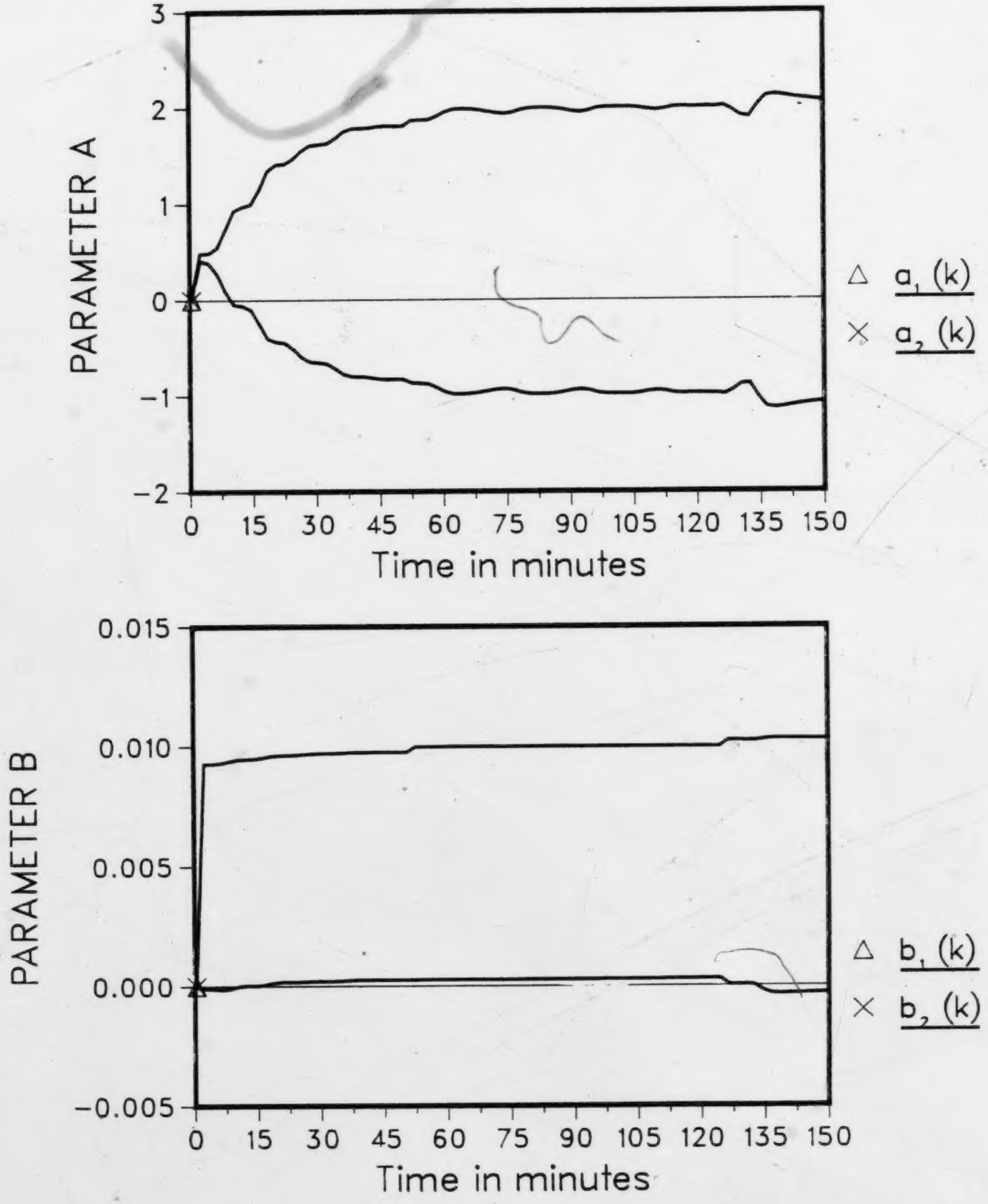


Fig.–5.25 Parameter trajectory of STC
STC/DC/d=0/

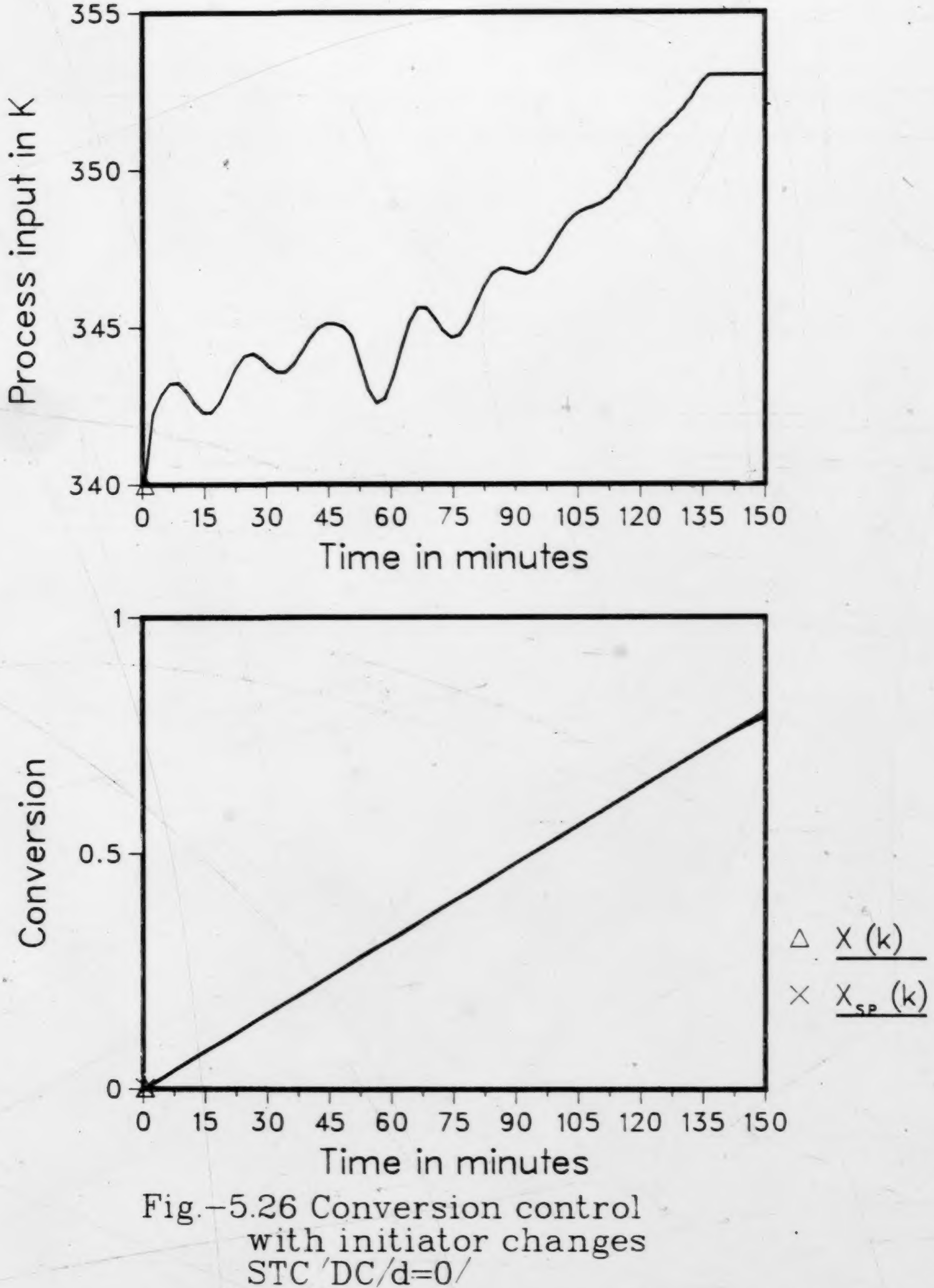


Fig.—5.26 Conversion control
with initiator changes
STC /DC/d=0/

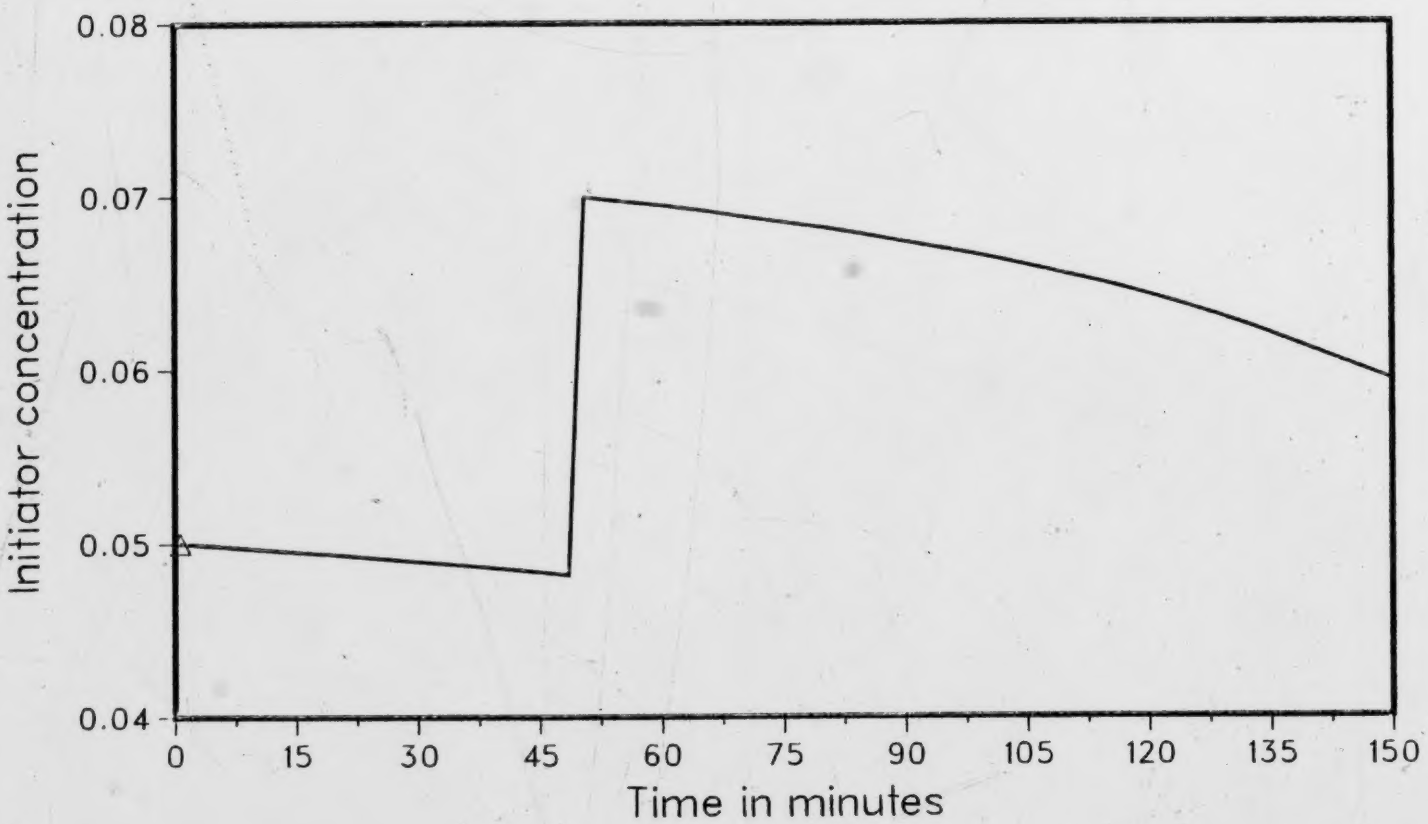


Fig. -5.27 Trajectory of initiator concentration with initiator addition at t=50 min under STC control .

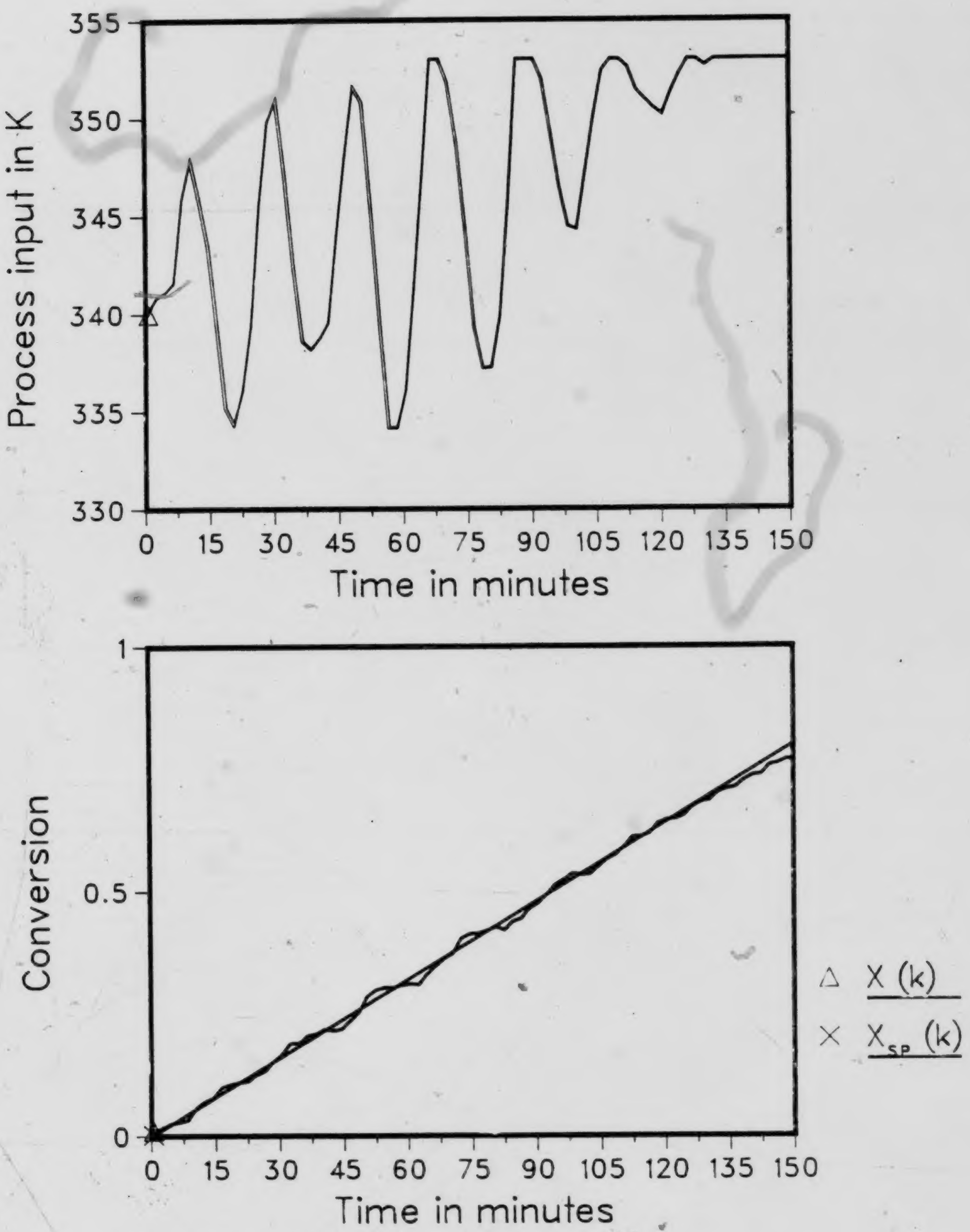


Fig.–5.28 Conversion control without set point changes STC/SC.005 d=0

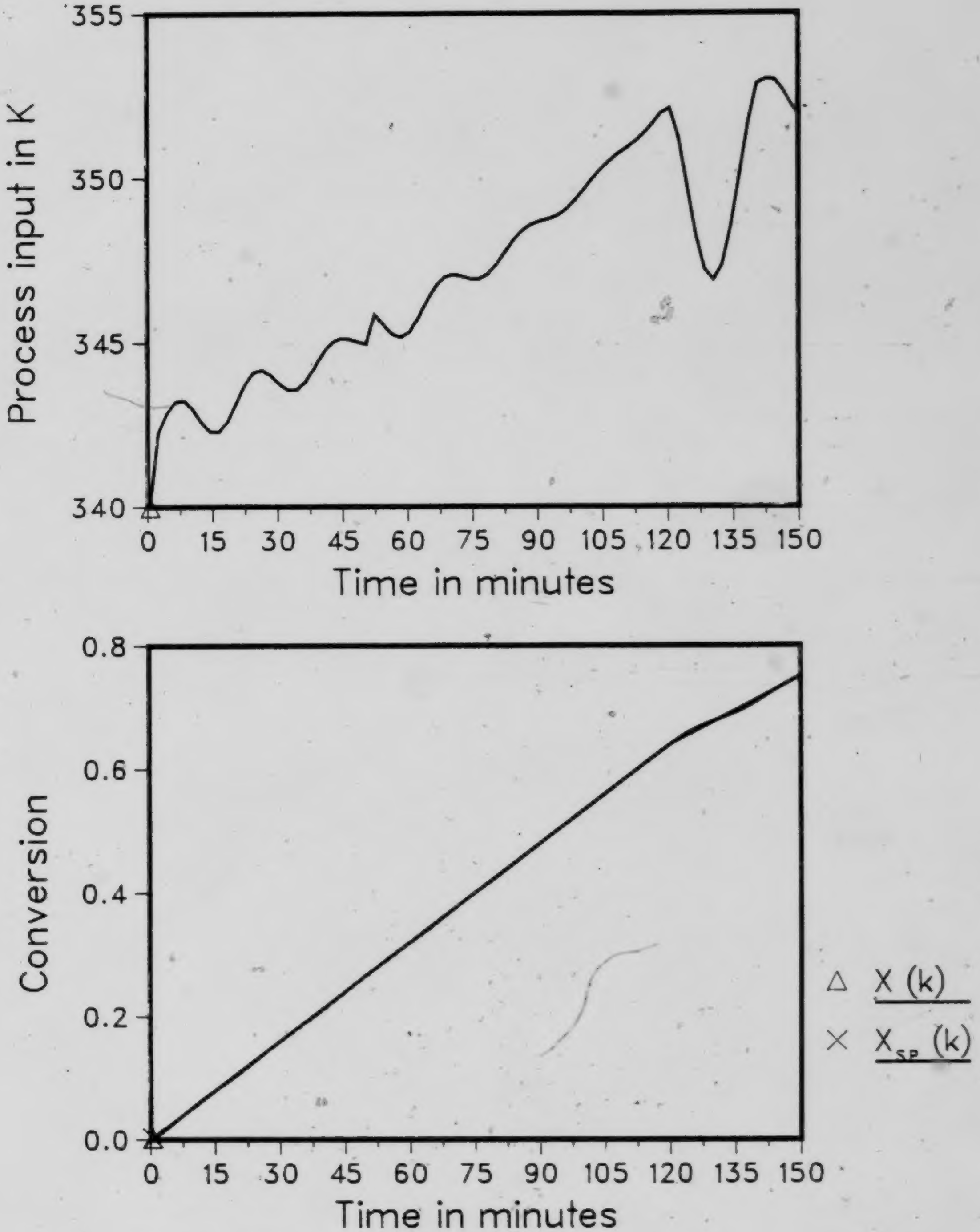


Fig.–5.29 Conversion control
with one set point change
STC/DC/d=0

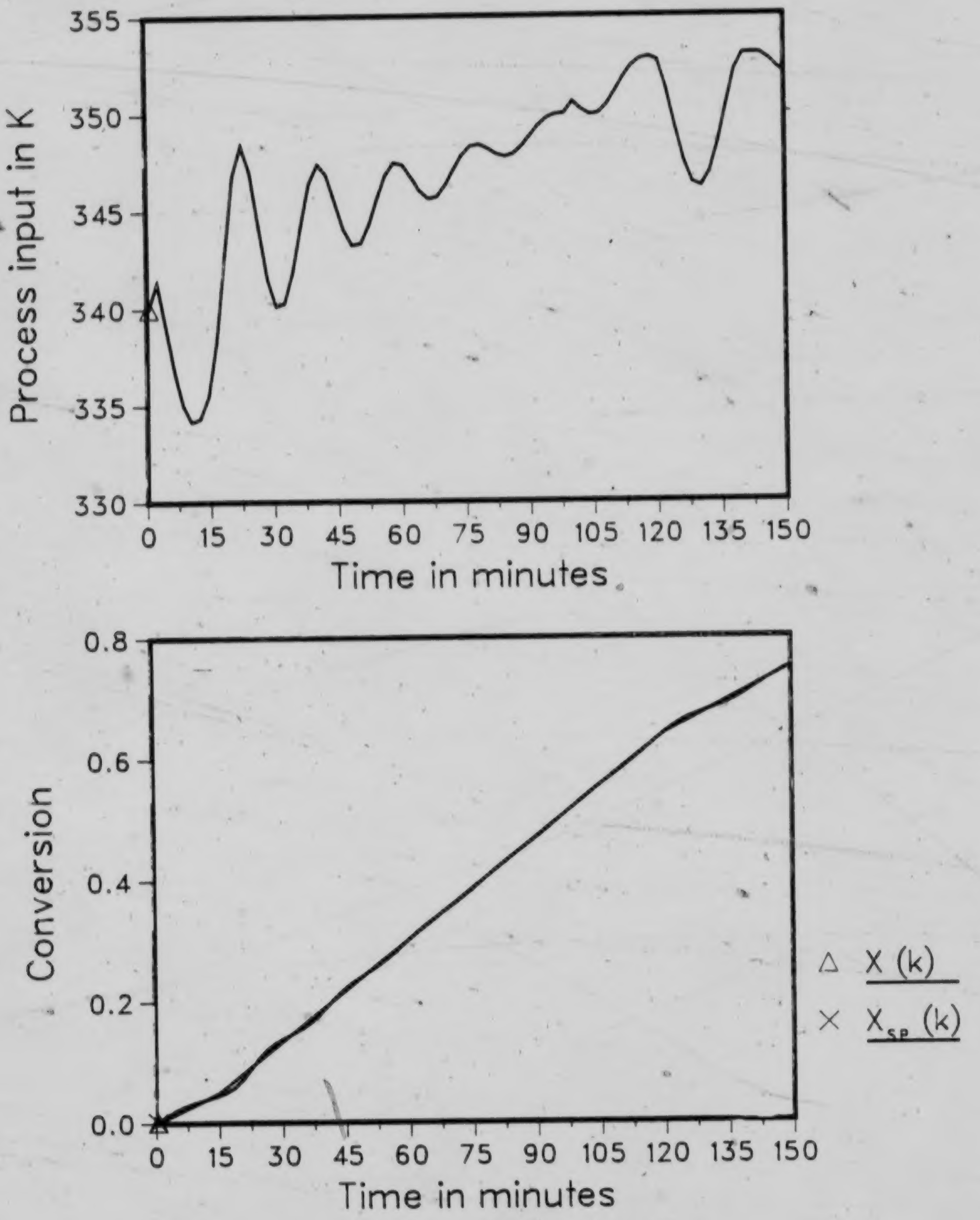


Fig.–5.30 Conversion control
with two set point changes
STC/DC/d=0/

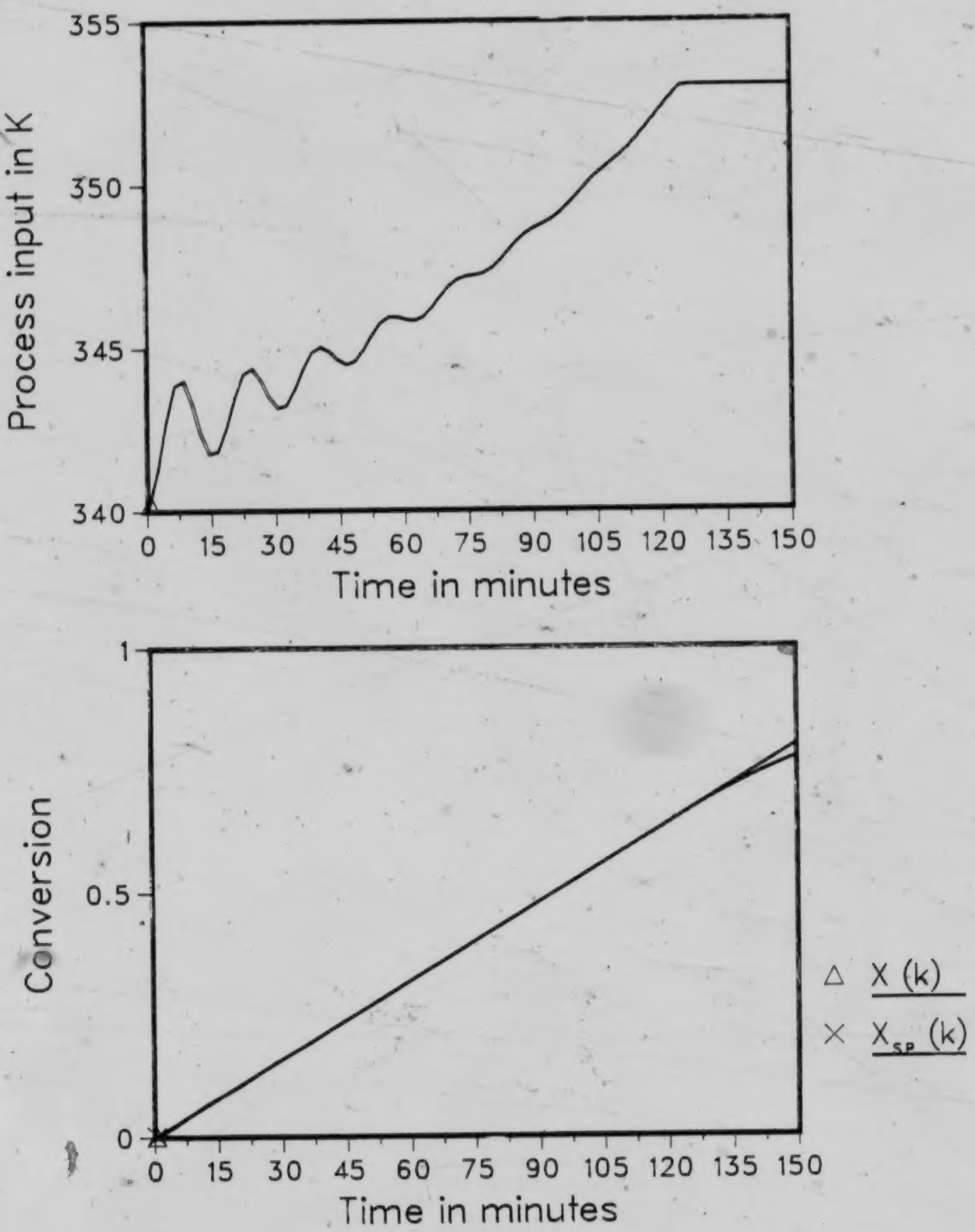


Fig.–5.31 Conversion control
with initiator changes
Fixed gain PID , DC.

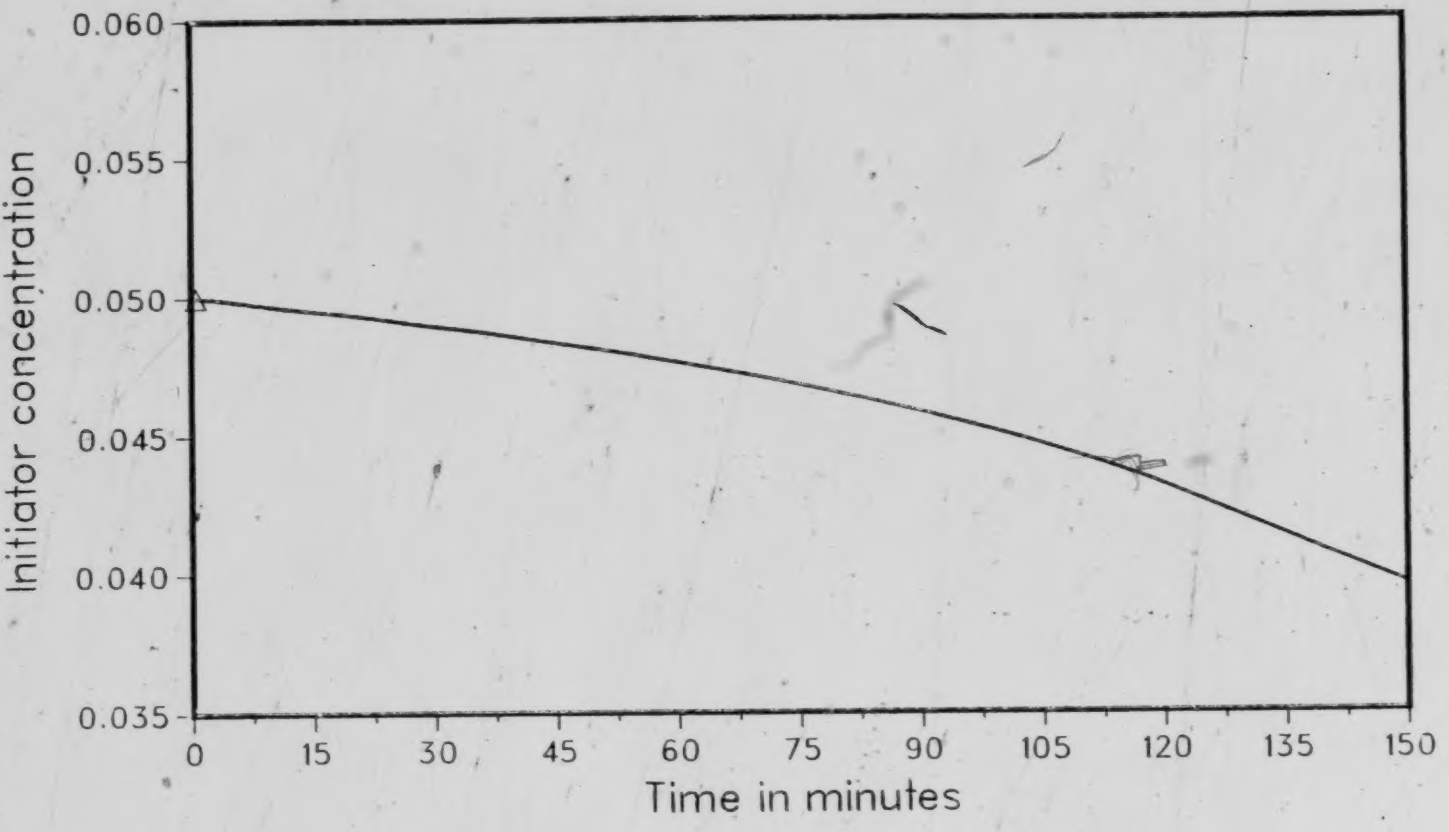


Fig. -5.32 Trajectory of initiator concentration under fixed gain PID control

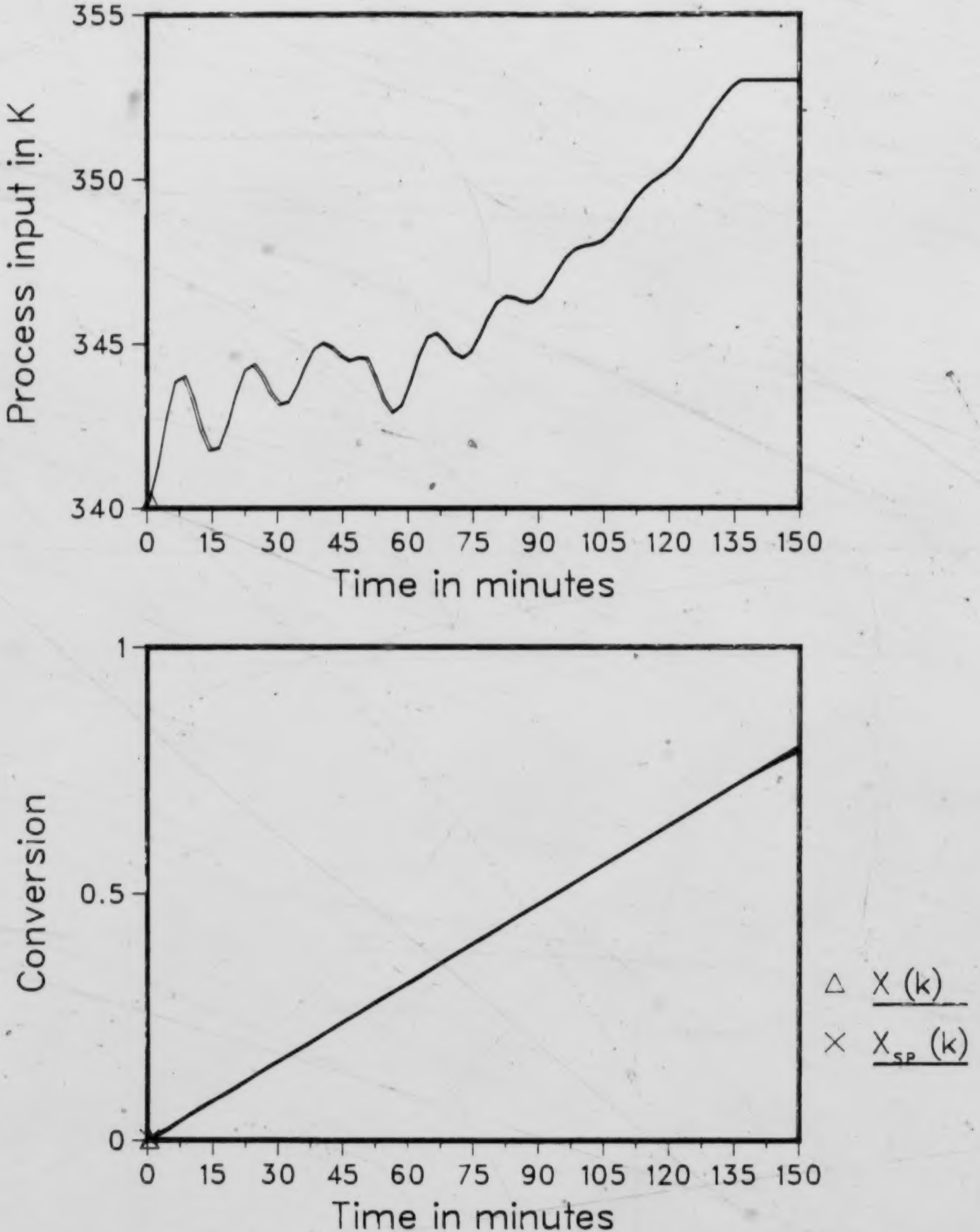


Fig.—5.33 Conversion control
without set point changes
Fixed gain PID , DC.

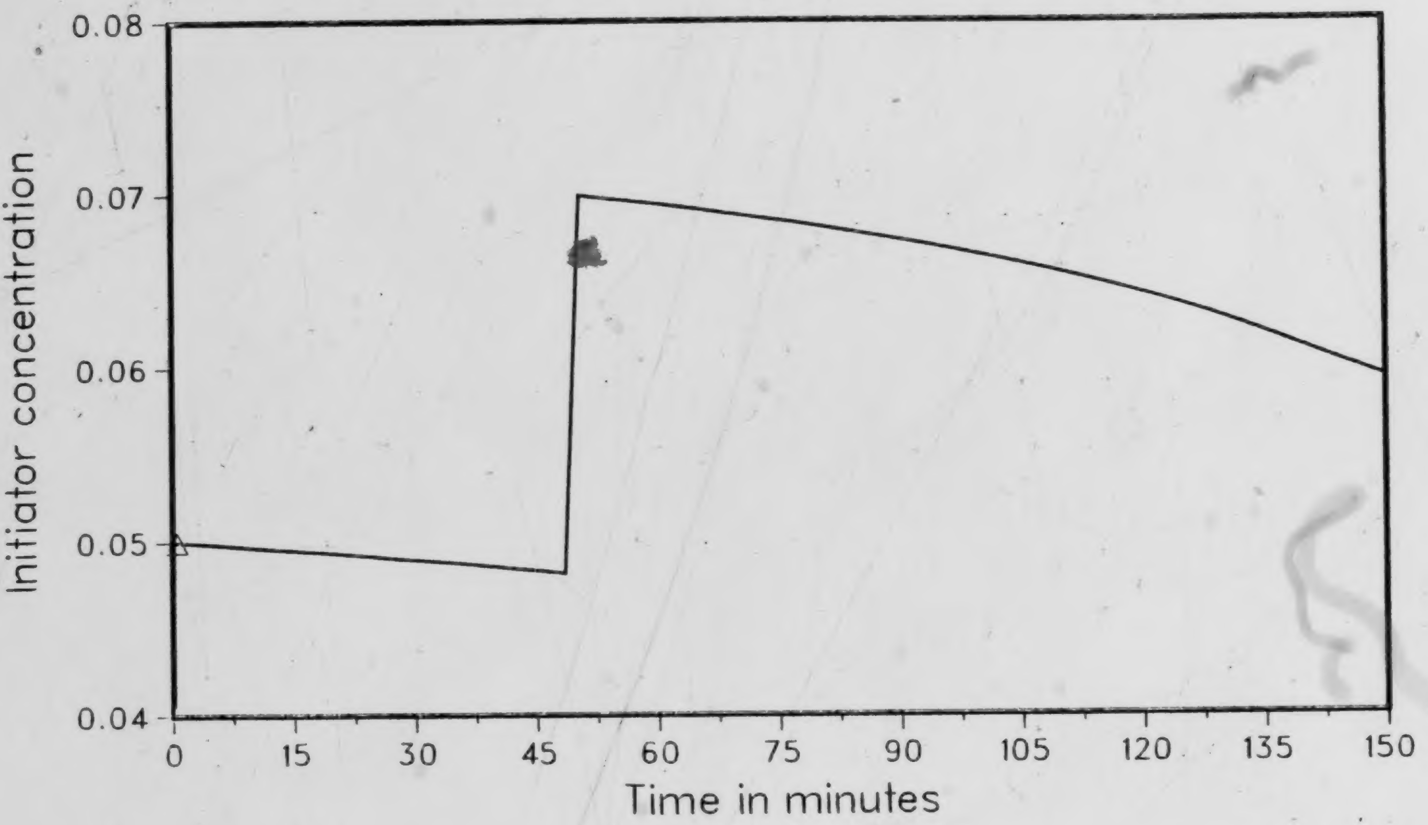


Fig -5.34 Trajectory of initiator concentration with initiator addition at t=50 min under fixed gain PID control.

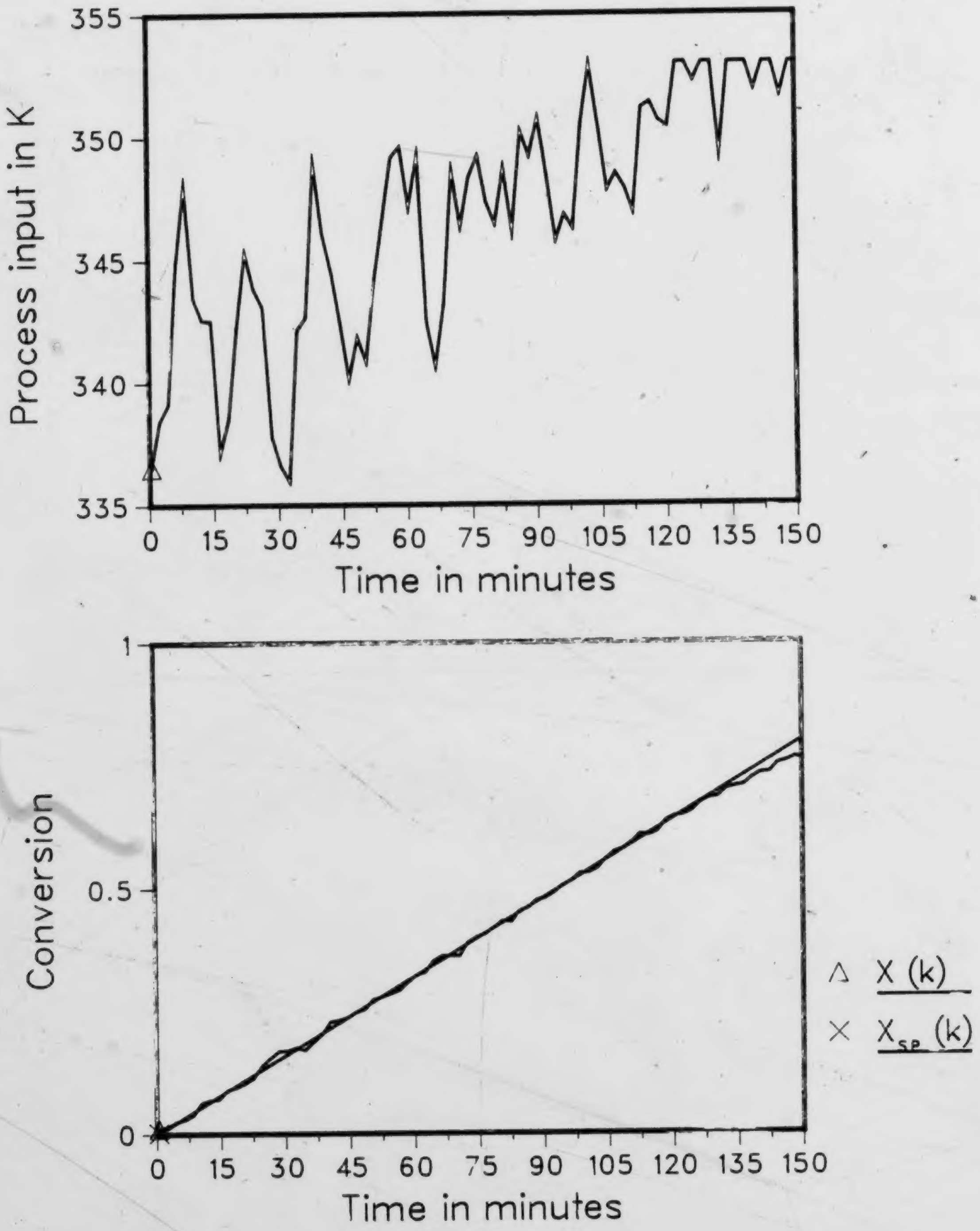


Fig.–5.35 Conversion control
without set point changes
Fixed gain PID , SC.005 .

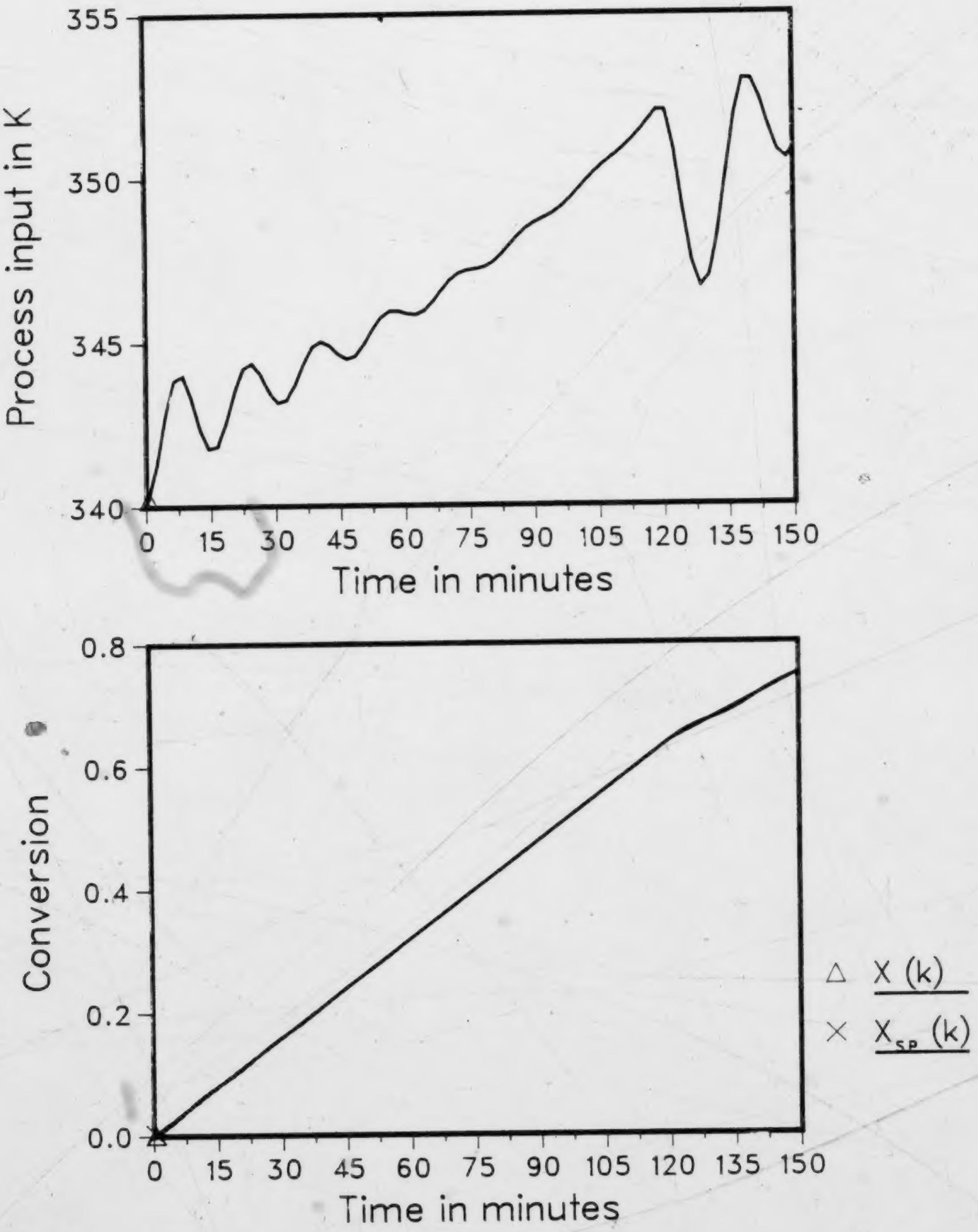


Fig.–5.36 Conversion control with one set point change Fixed gain PID , DC.

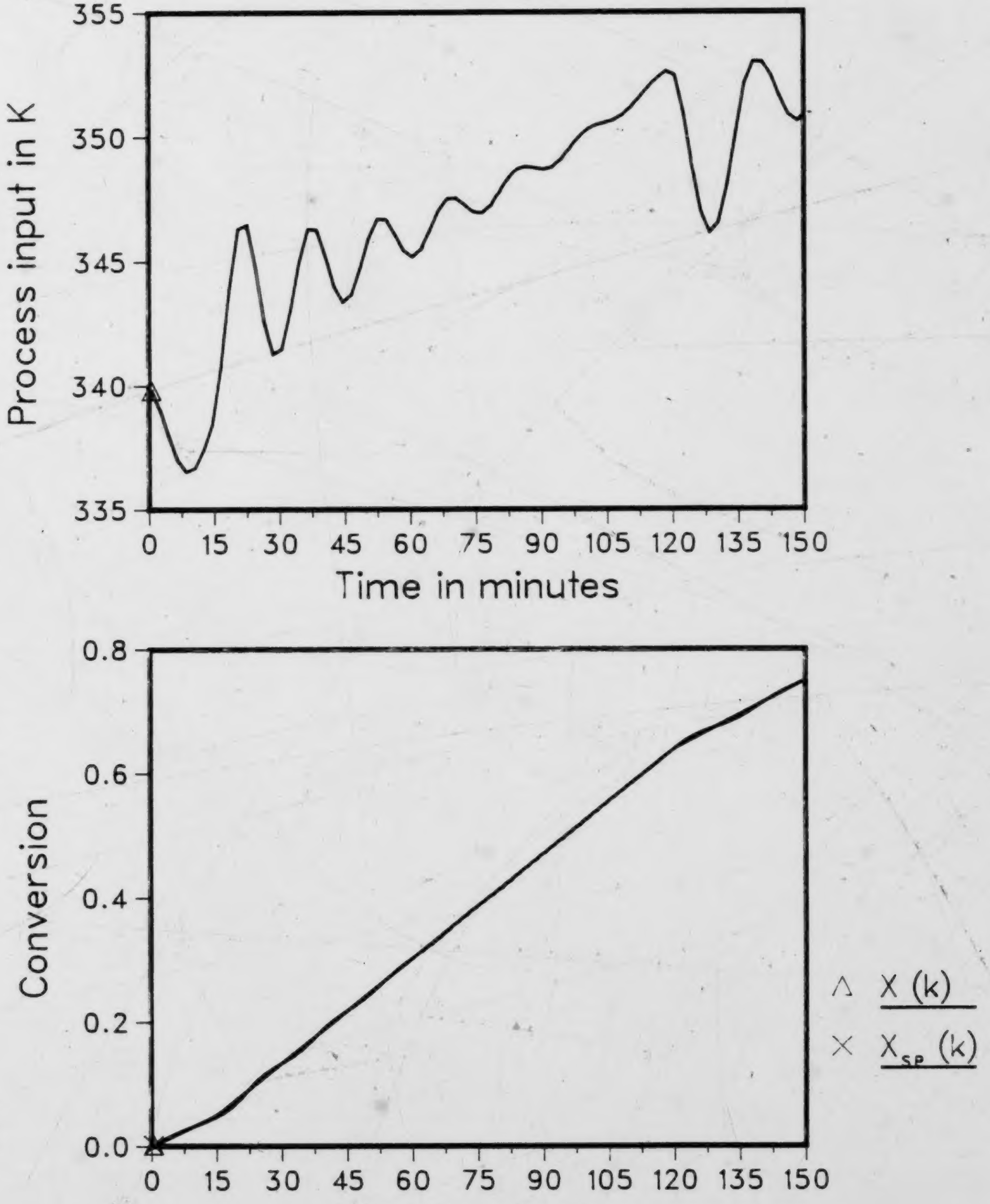


Fig.–5.37 Conversion control with two set point changes Fixed gain PID , DC.

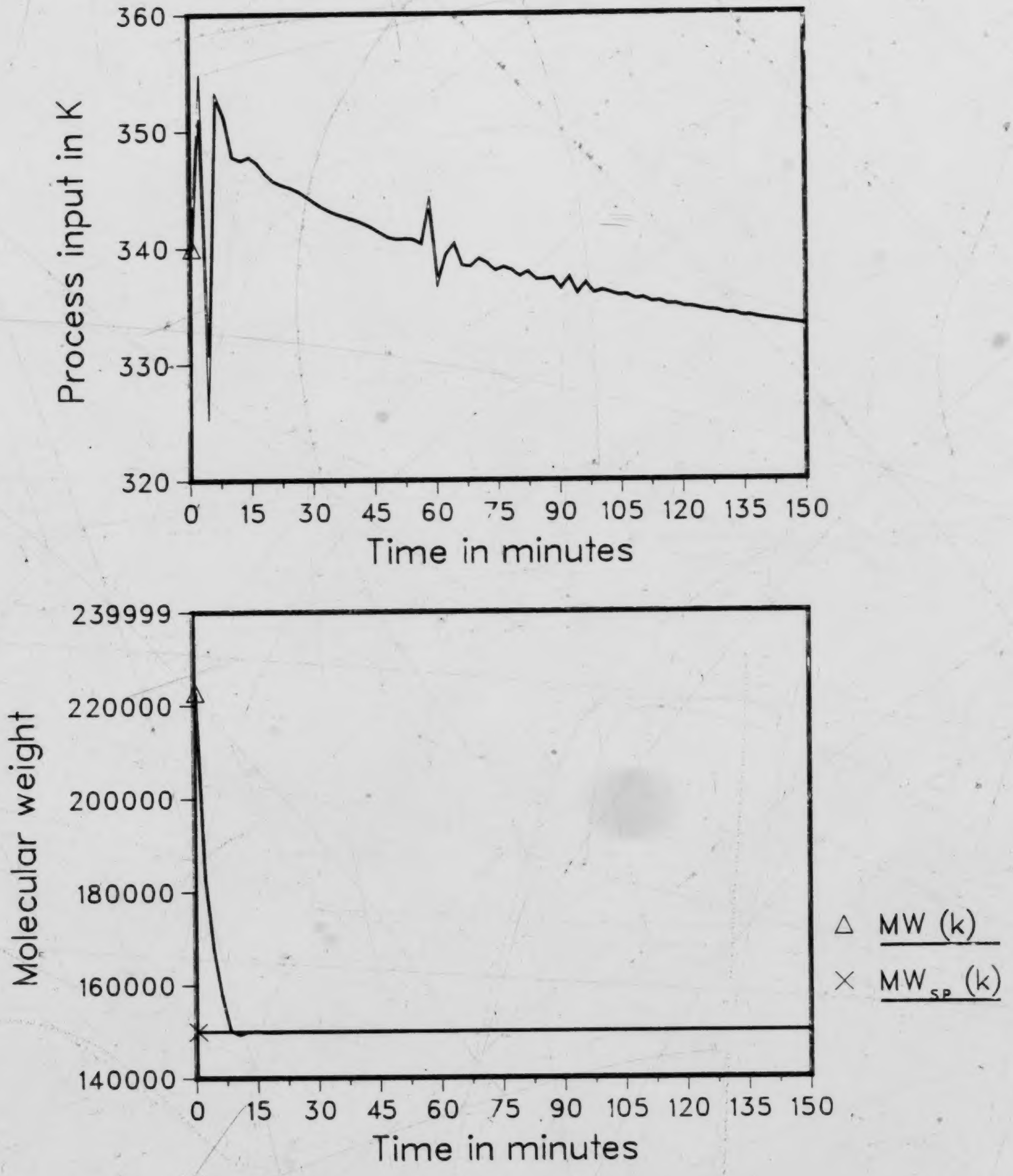


Fig.–5.38 Molecular weight control
PA/IA/DC/d=0/p=0.5/Uf=340

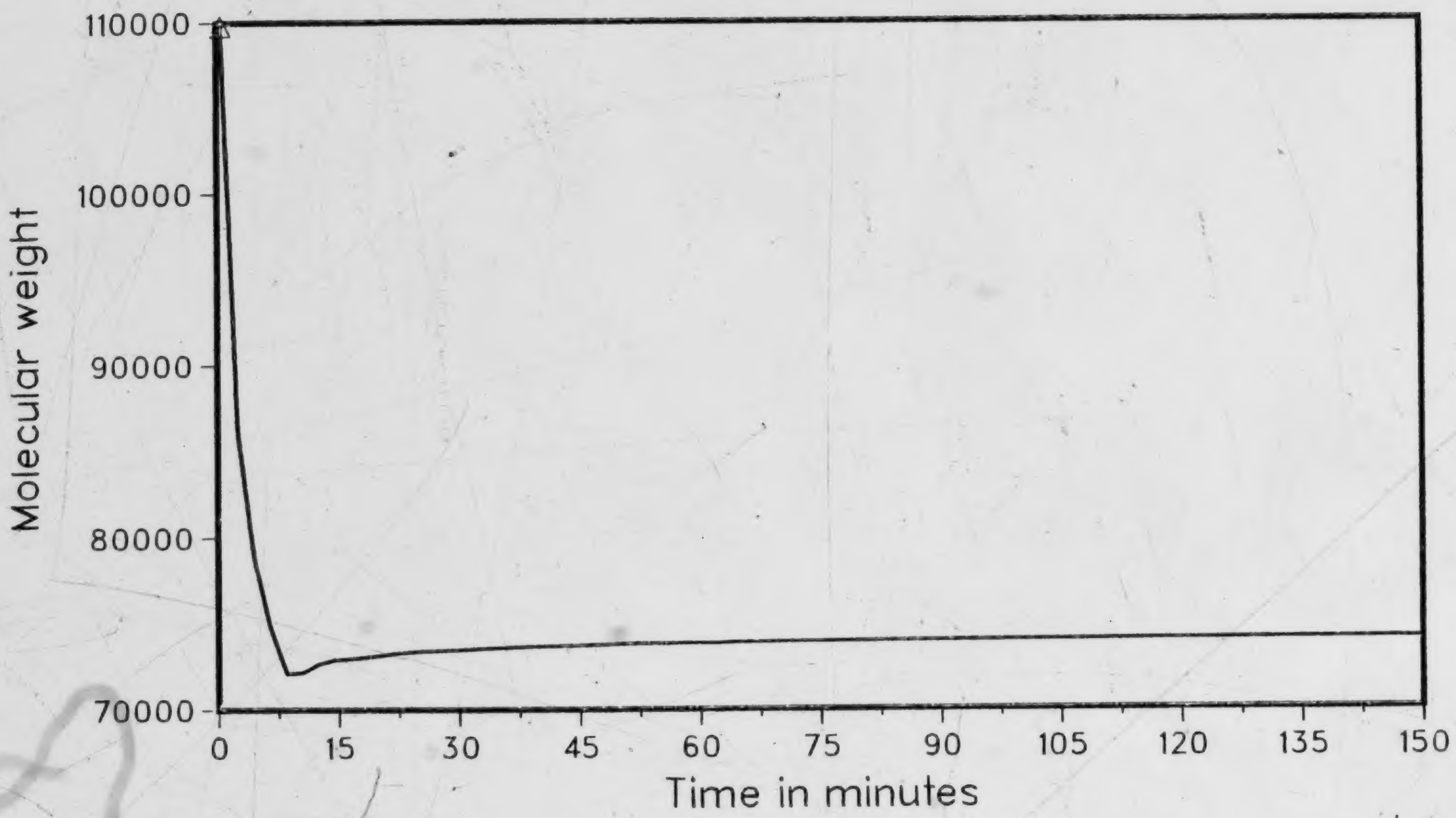


Fig. -5.39 Number average molecular weight trajectory
(cf. Fig. -5.38)

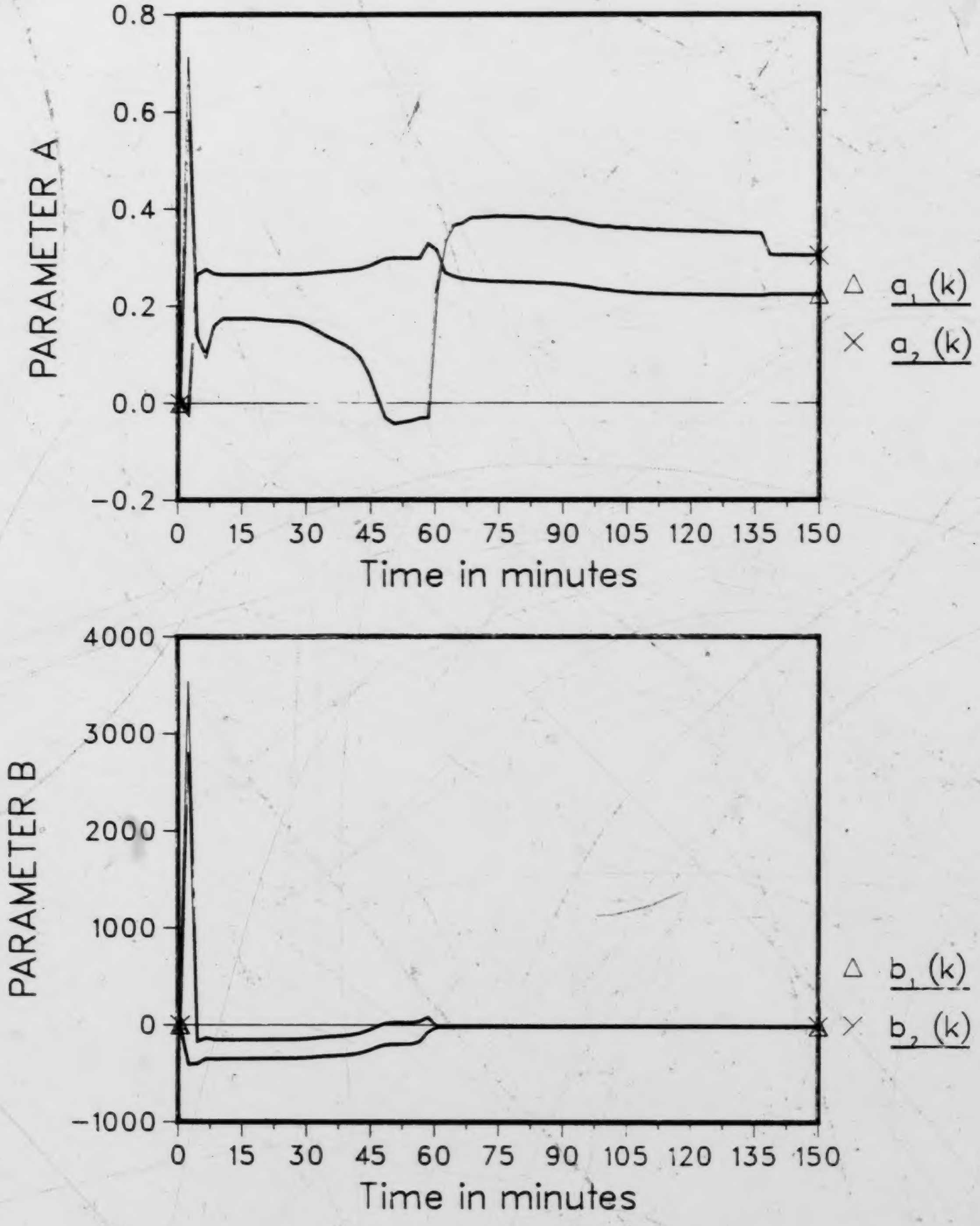


Fig.–5.40 Parameter trajectory of PA
PA/IA/DC/d=0/p=0./Uf=340.

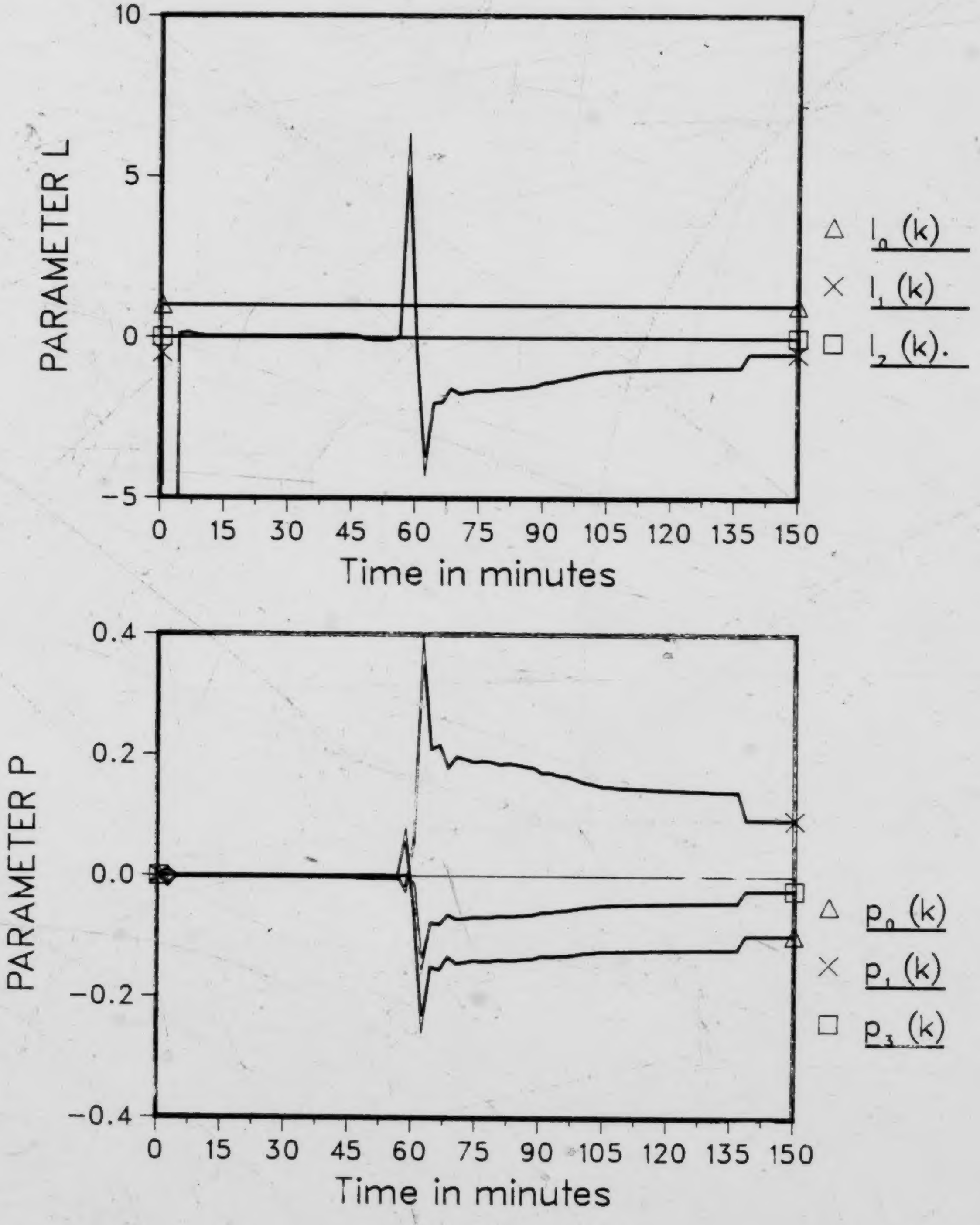


Fig.–5.41 Parameter trajectory of PA
PA /IA /DC /d=0/p=0.5 /Uf=340.

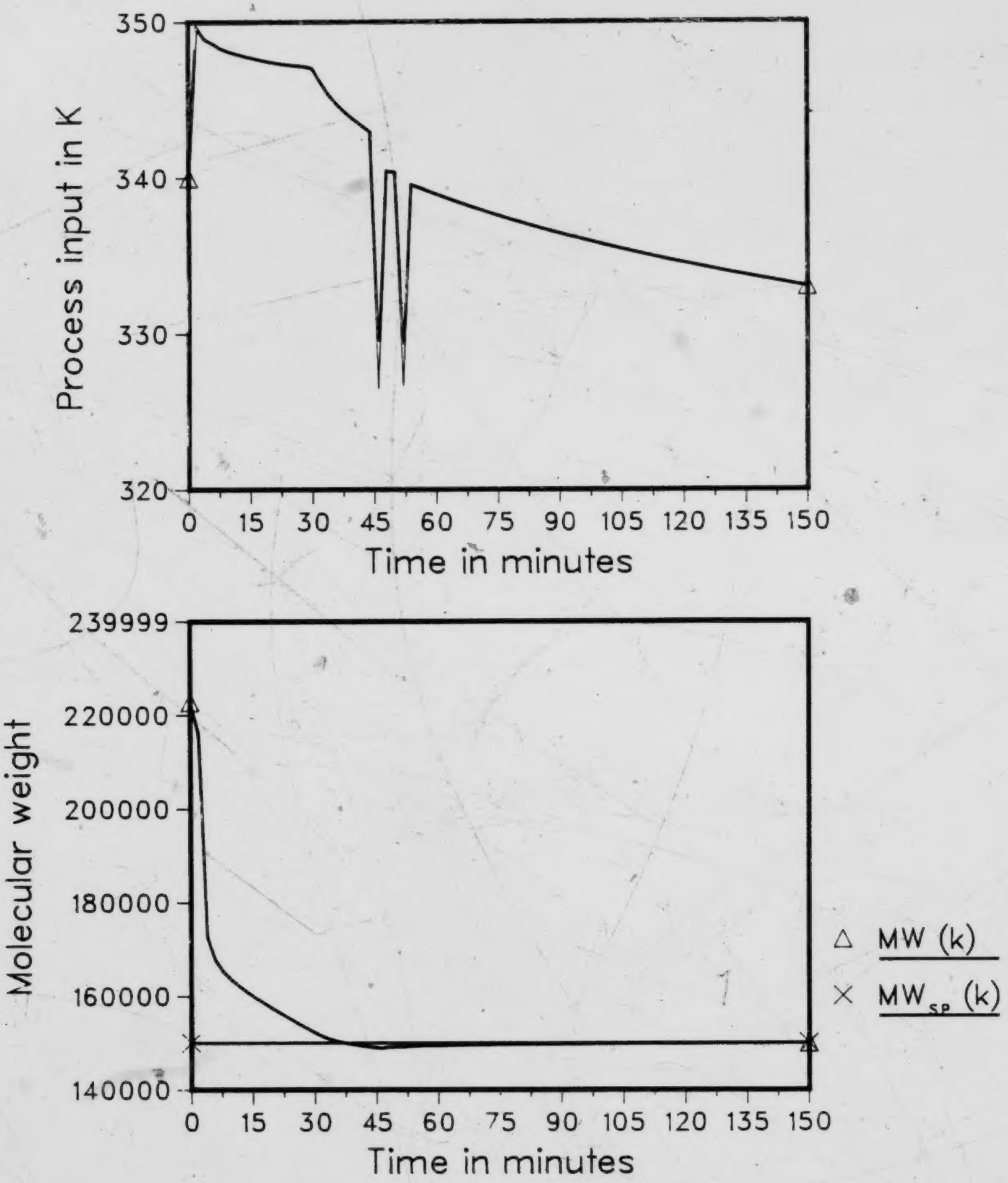


Fig.–5.**42** Molecular weight control PID/DC/d=0/p=0./Uf=340.

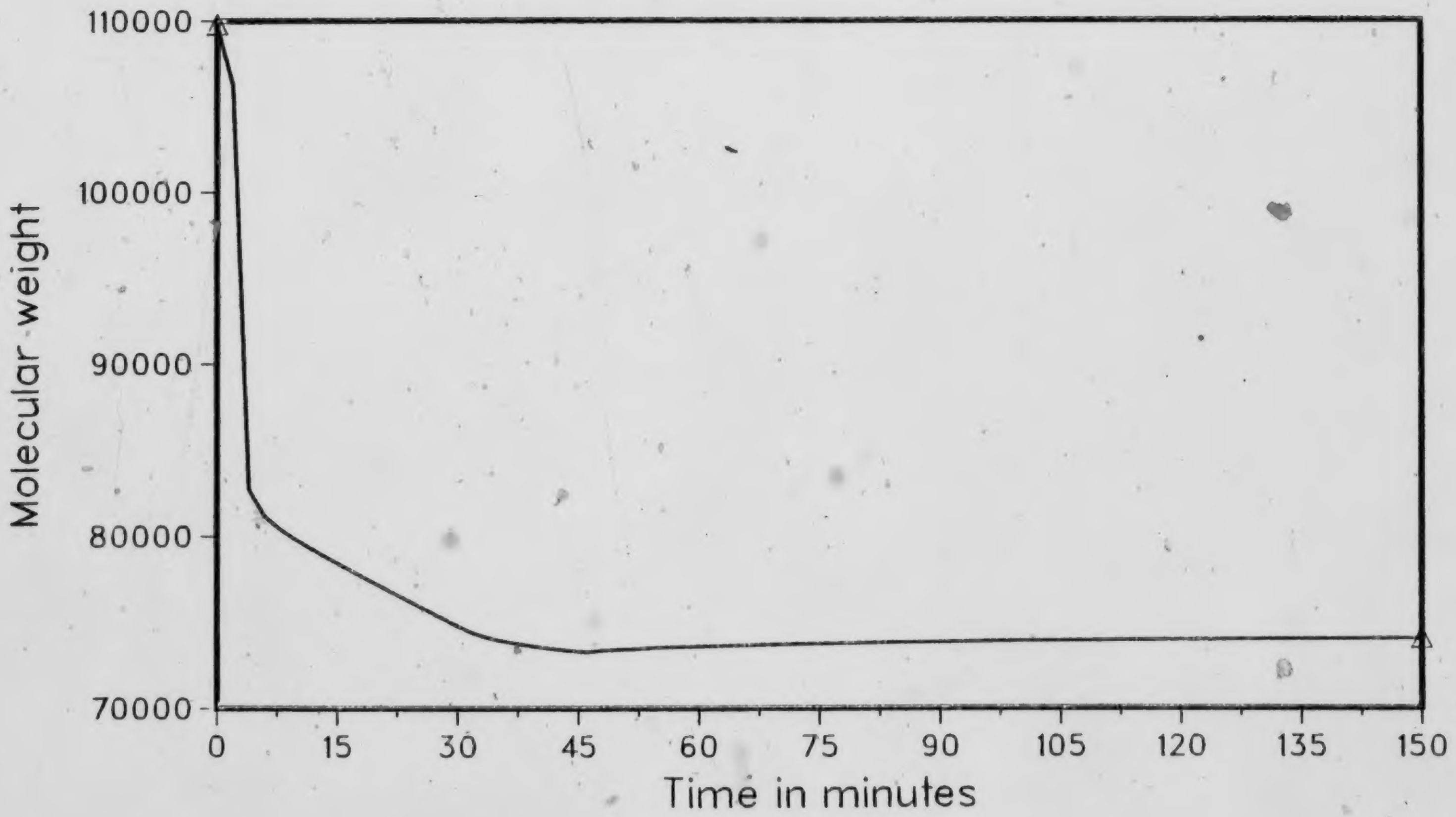


Fig. -5.**43** Number average molecular weight trajectory
(cf. Fig. -5.**42**)

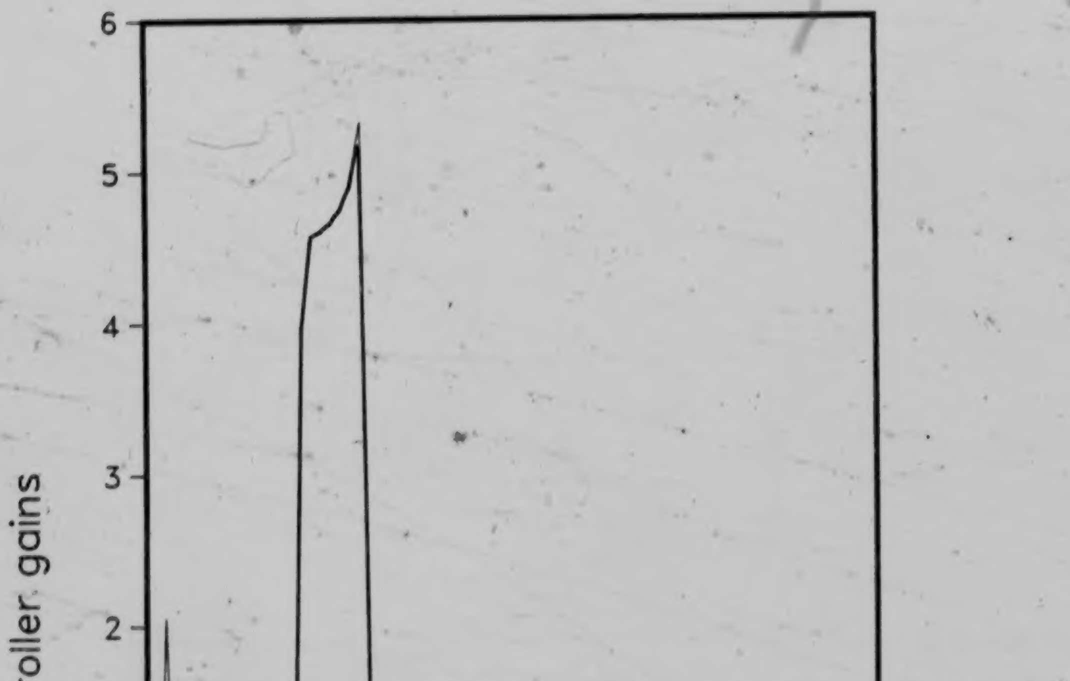


Fig.–5.44 Parameter trajectory of PID
PID/DC/d=0/p=0./Uf=340.

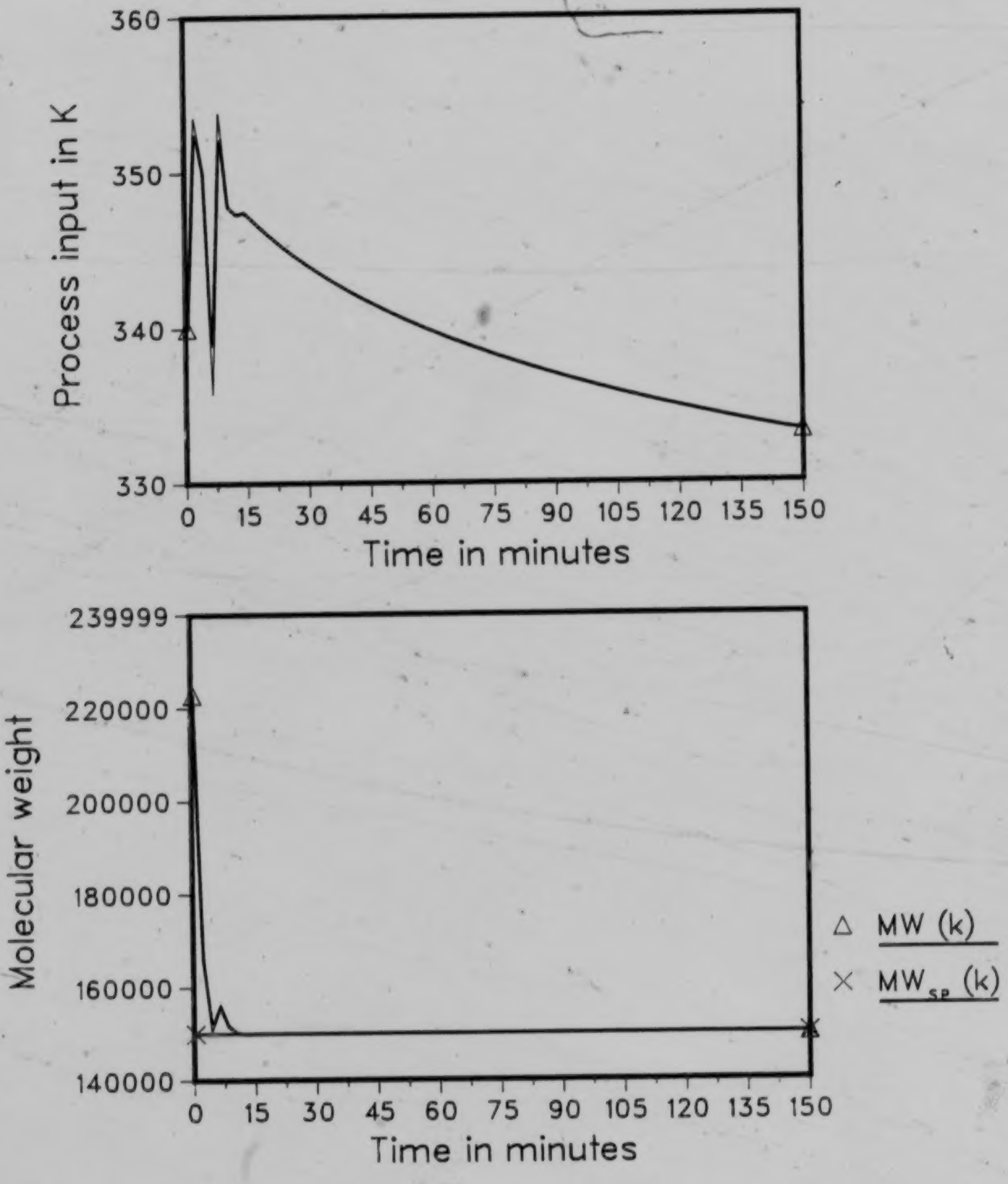


Fig.–5.**45** Molecular weight control with an STR , DC.

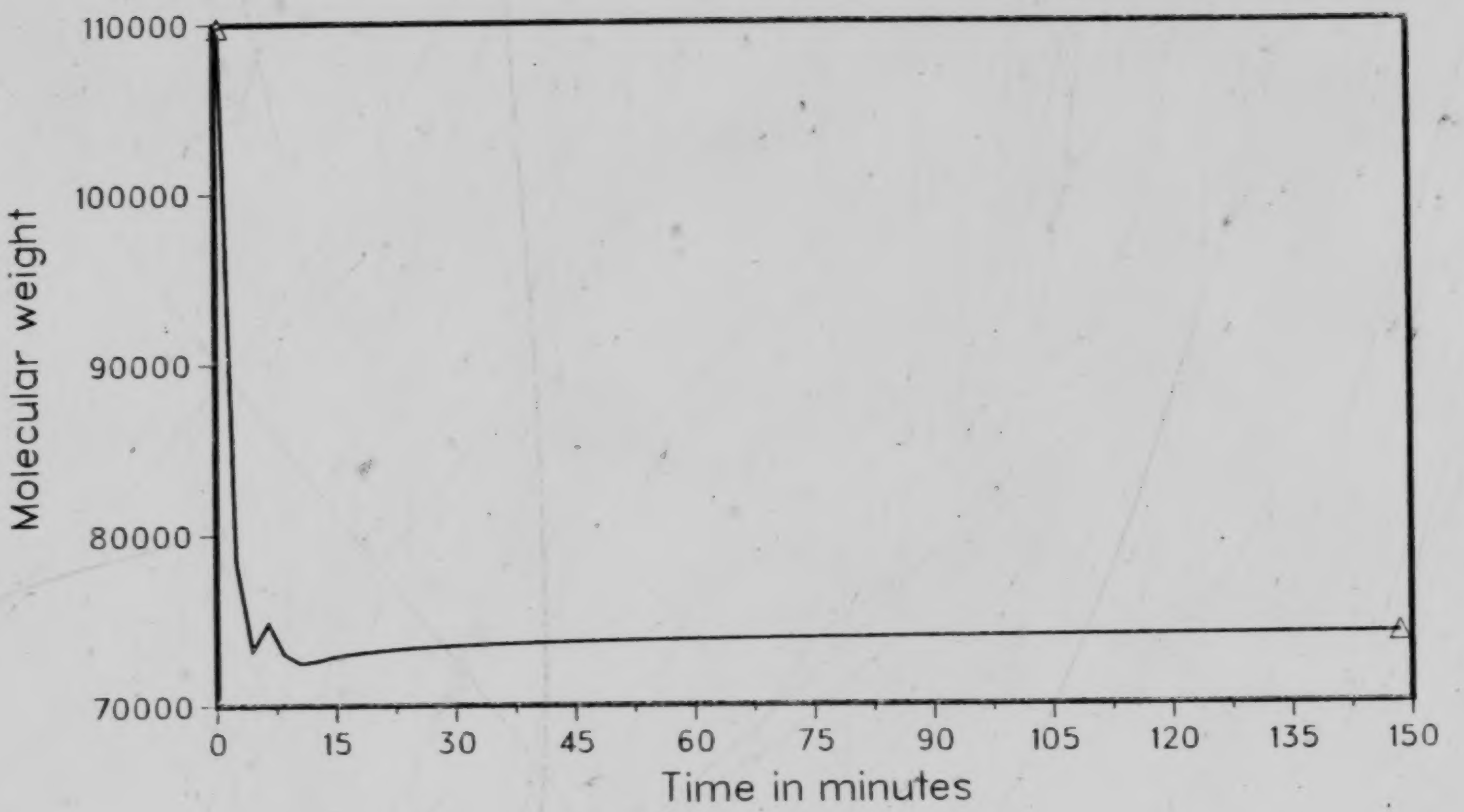


Fig. -5.**46** Number average molecular weight trajectory
(cf. Fig. -5.**45**)

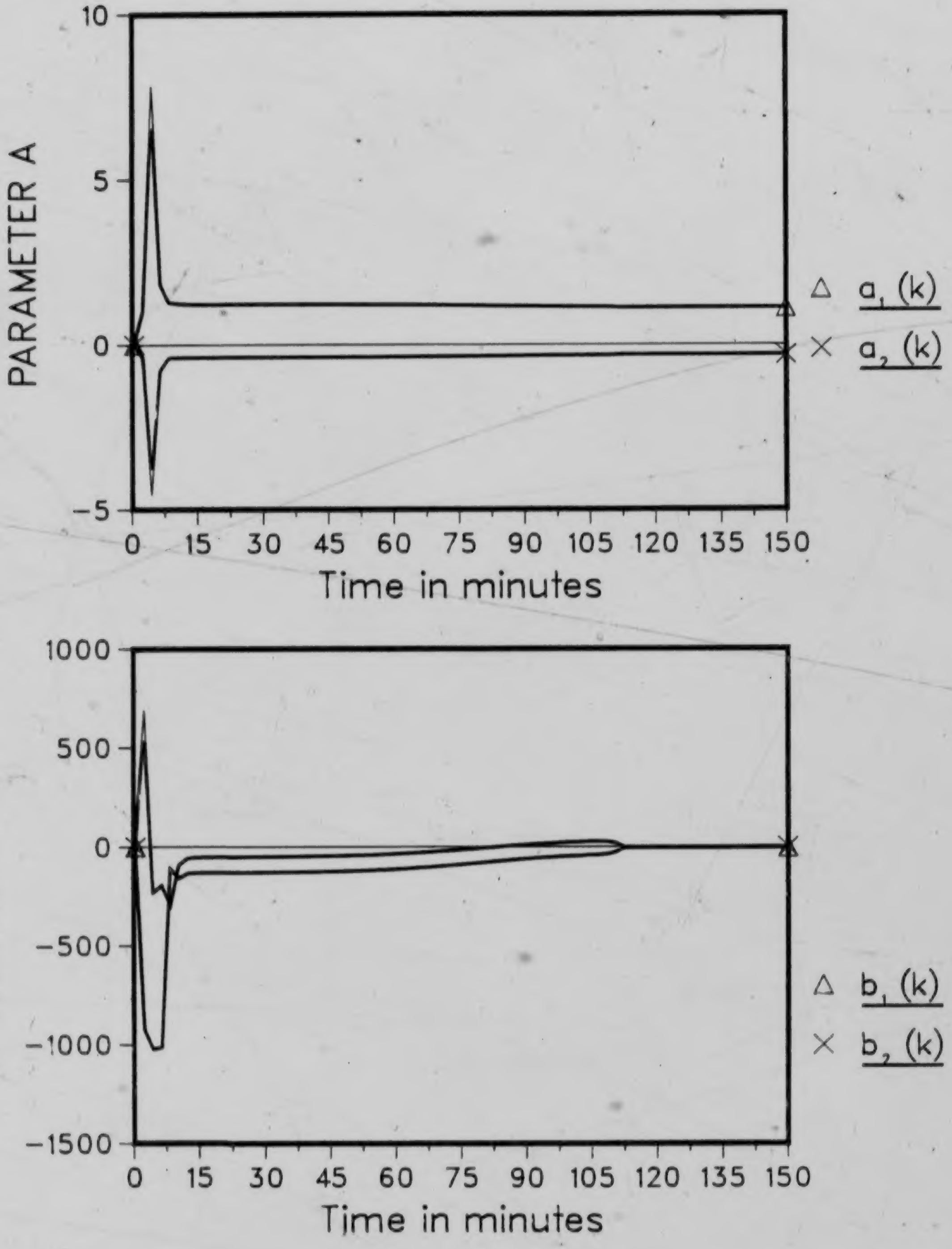


Fig.−5.**47** Parameter trajectory of STC

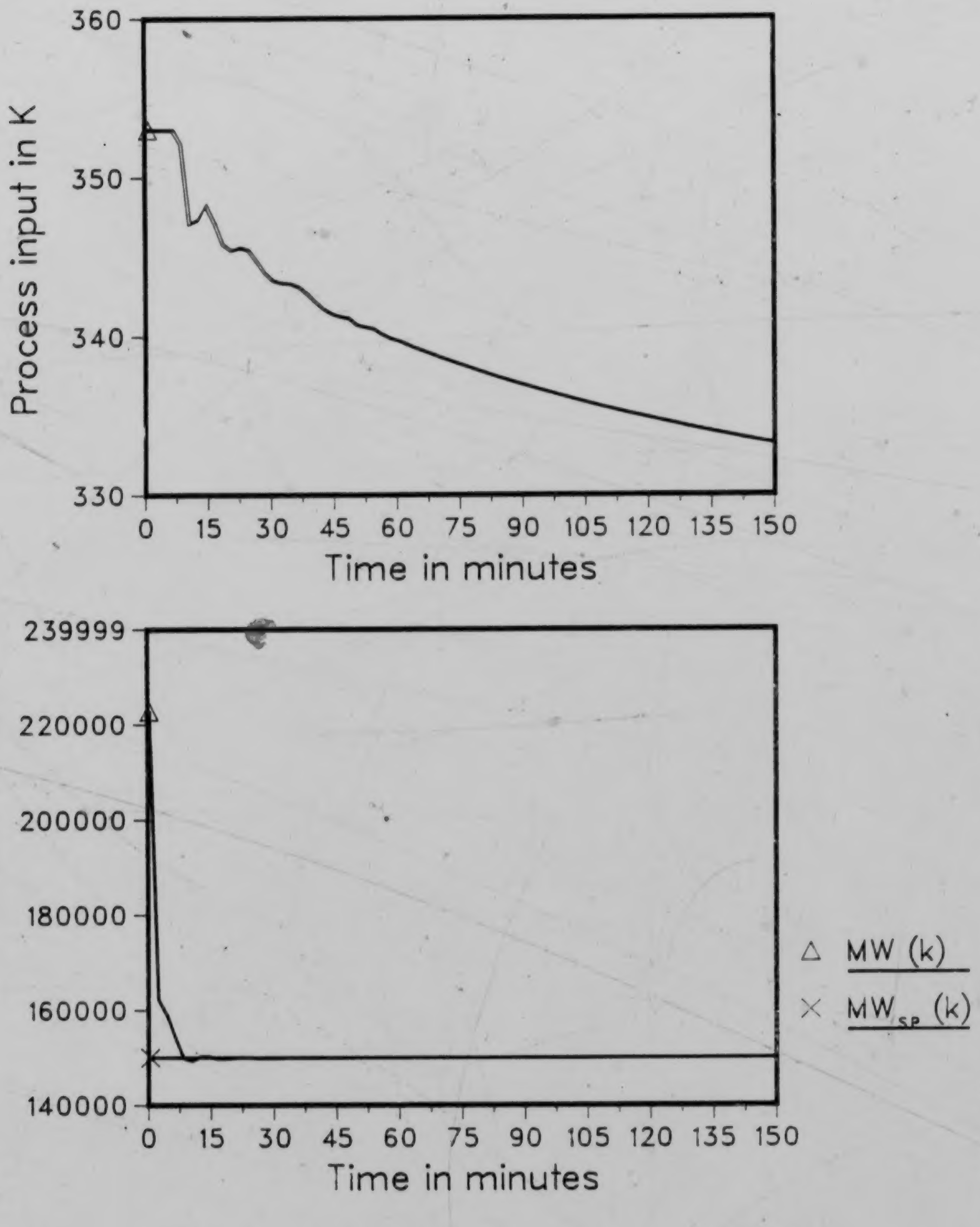


Fig.–5.**48** Molecular weight control
with a fixed gain PID.

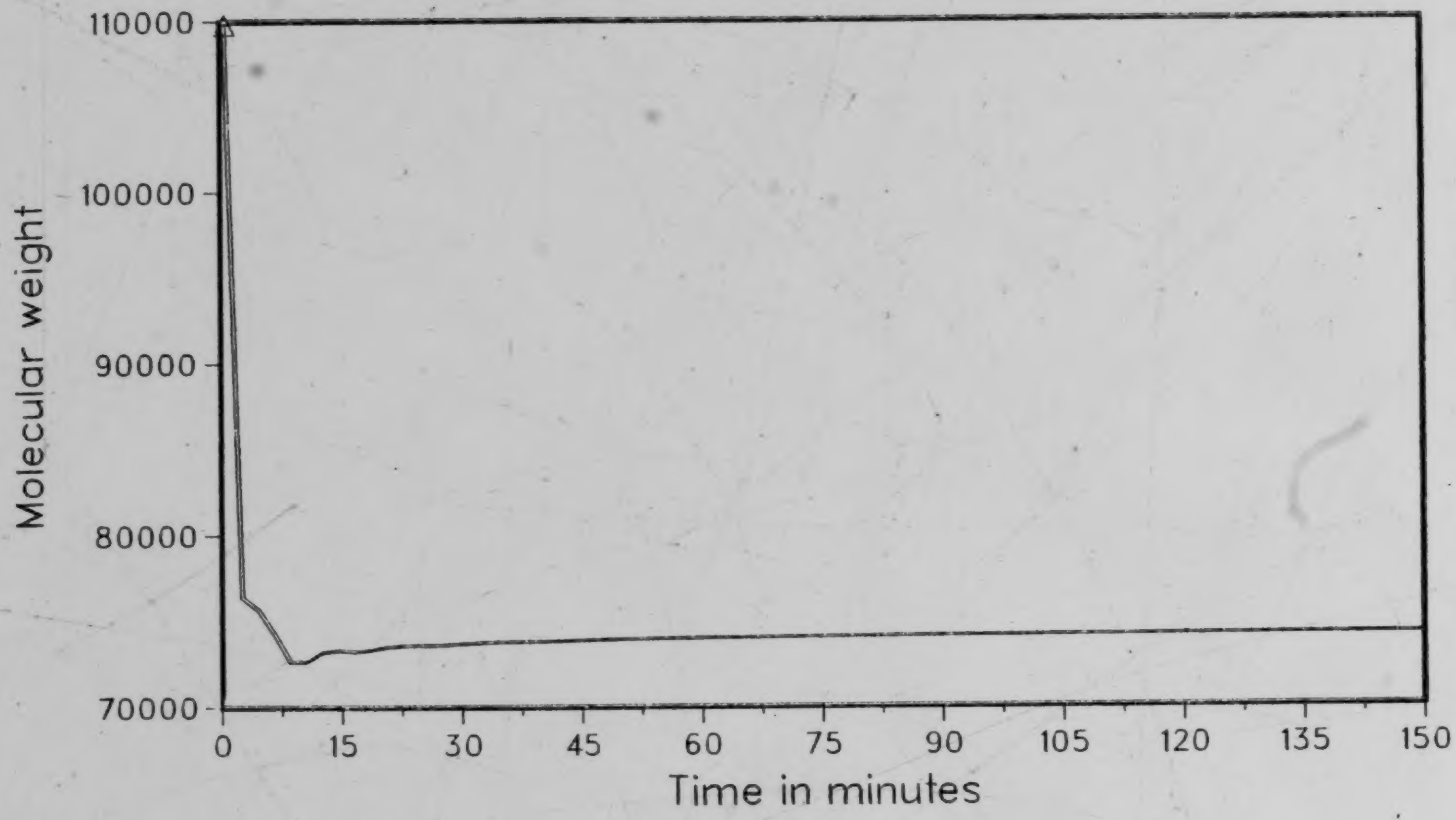


Fig. -5.**49** Number average molecular weight trajectory (cf. Fig. -5.**48**)

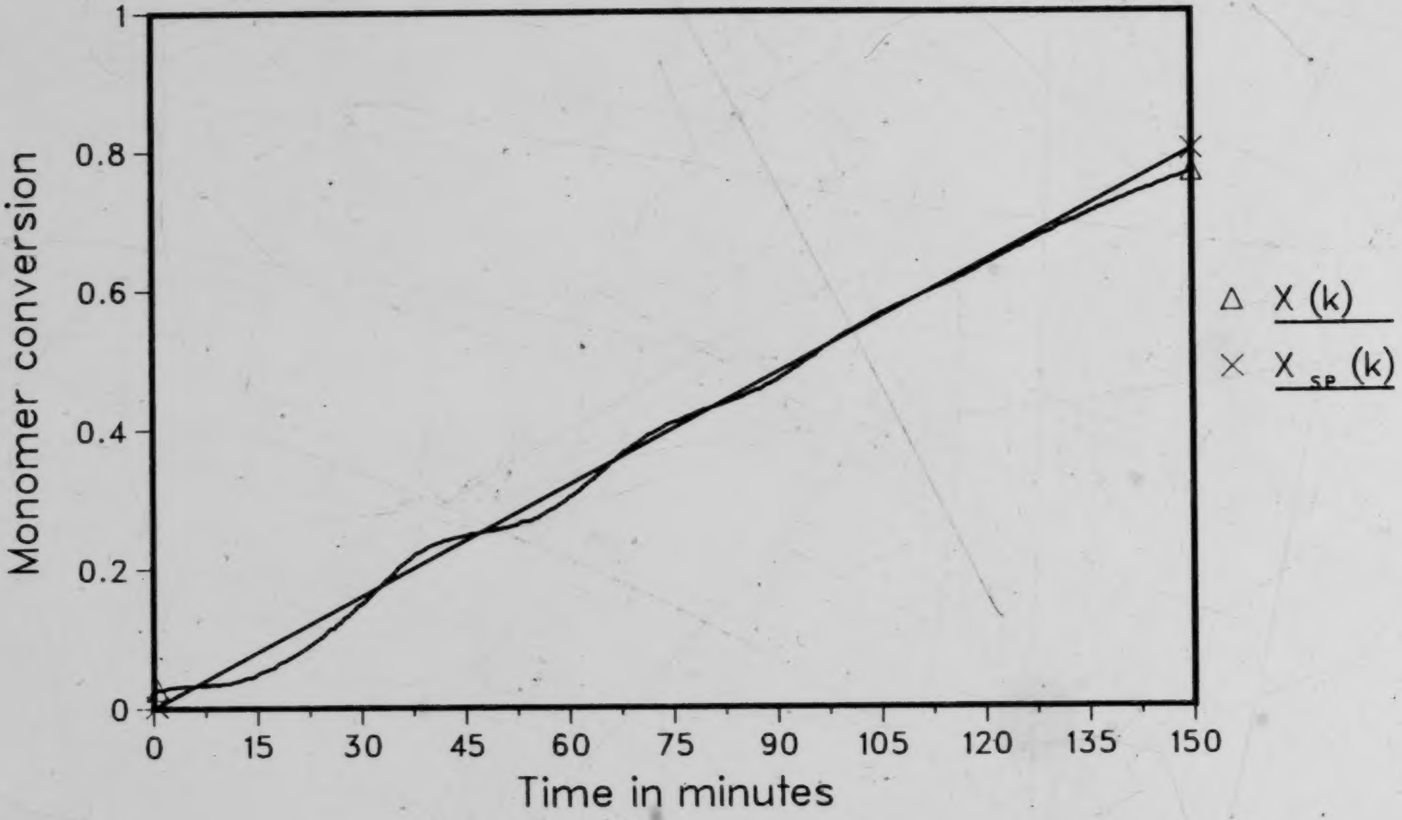


Fig.–5.50 Conversion control with PID

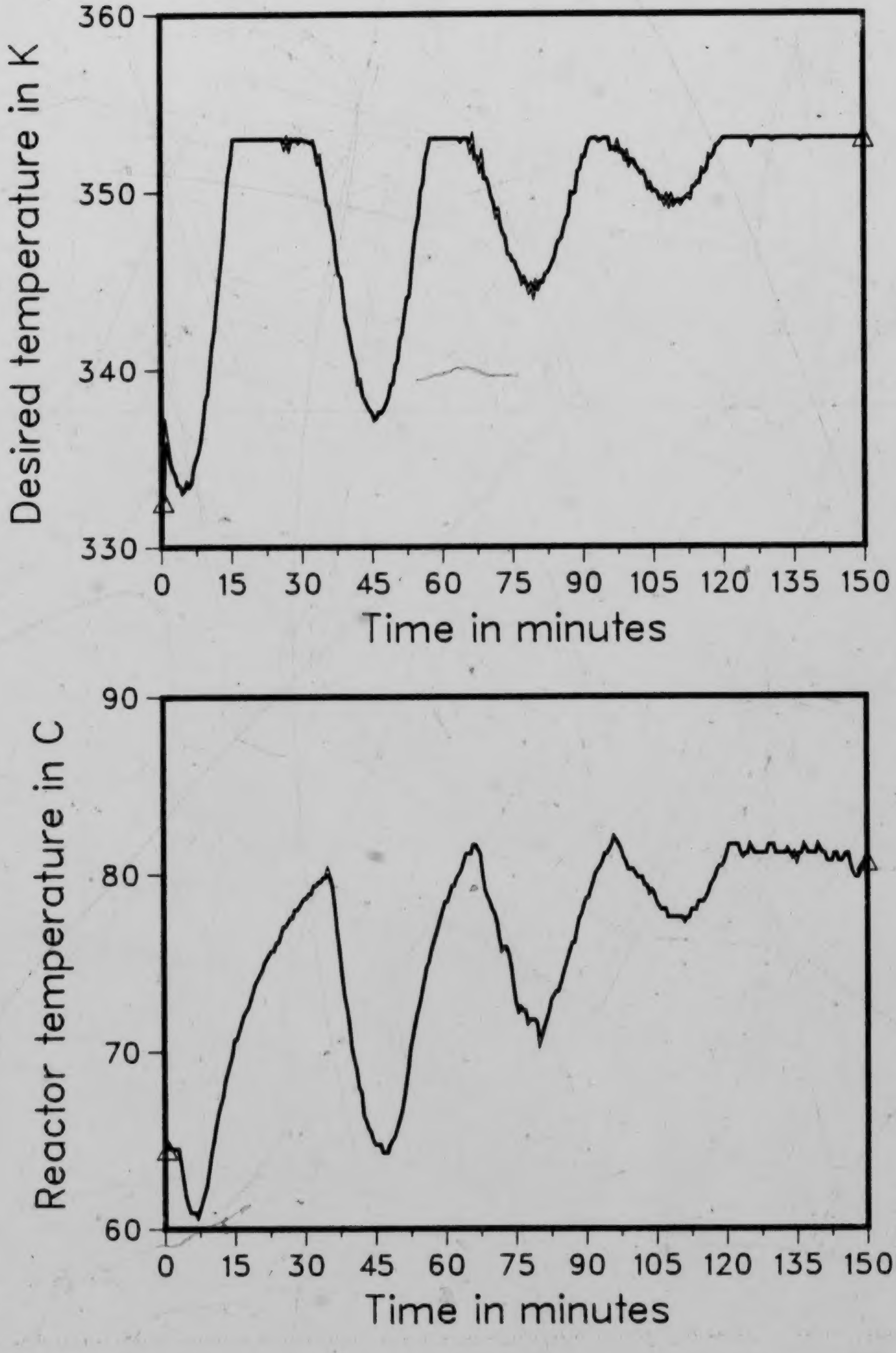


Fig.–5.51 Desired and actual reactor temperature (cf. Fig.–5.50)

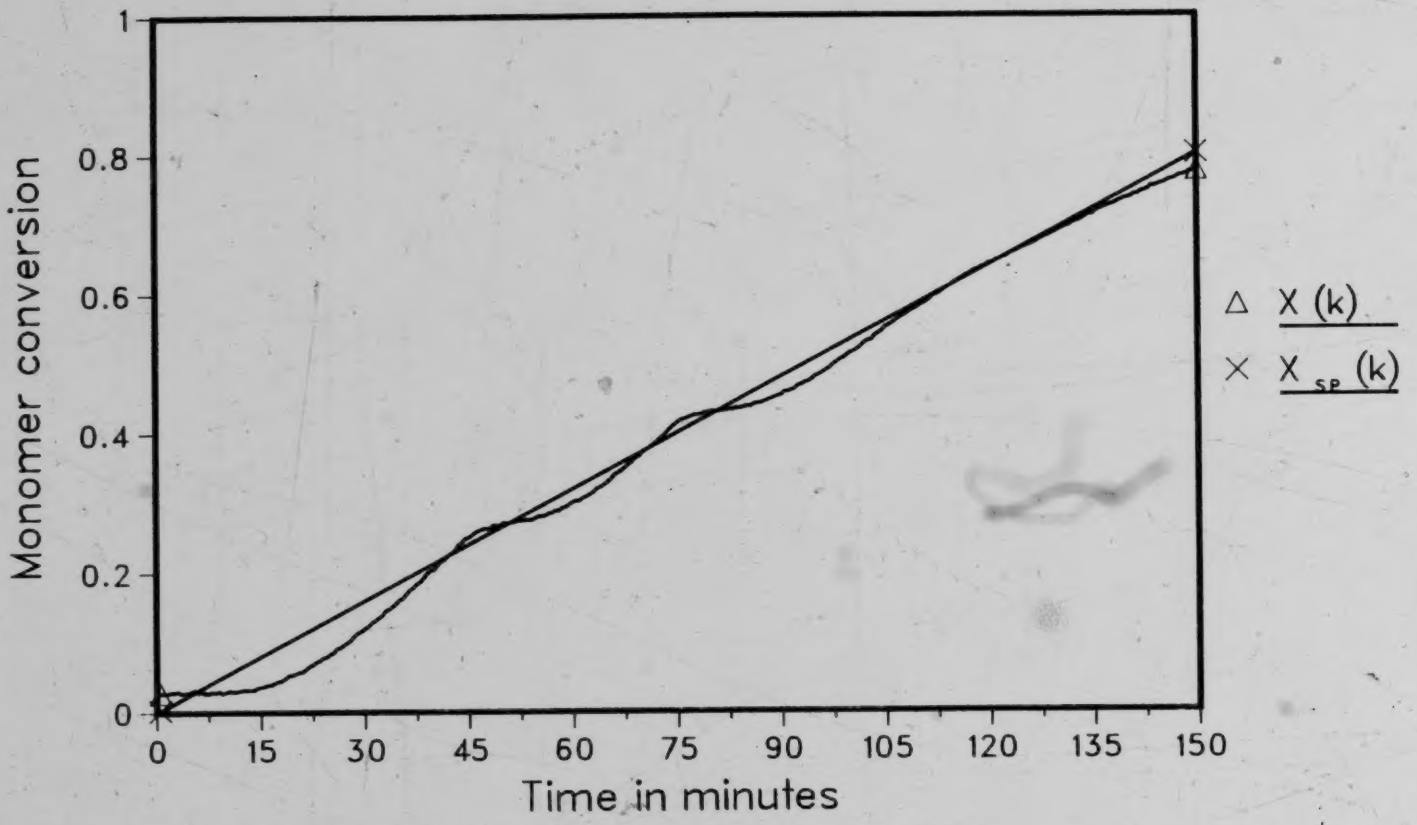


Fig.–5.52 Conversion control with STC
STC/na=3/nb=2/
(nonzero initial parameters)

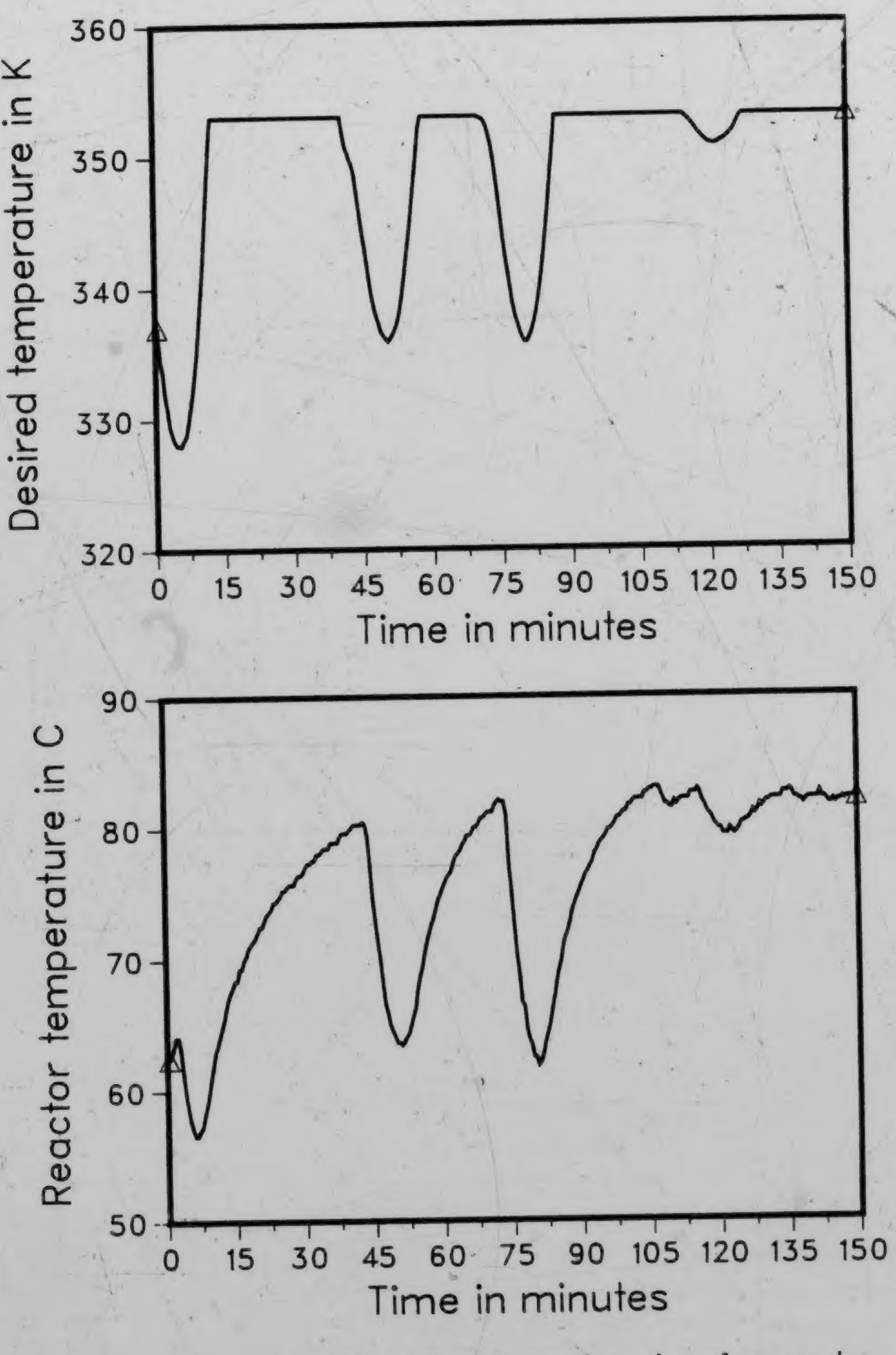


Fig.–5.53 Desired and actual reactor temperature (cf. Fig.–5.52)

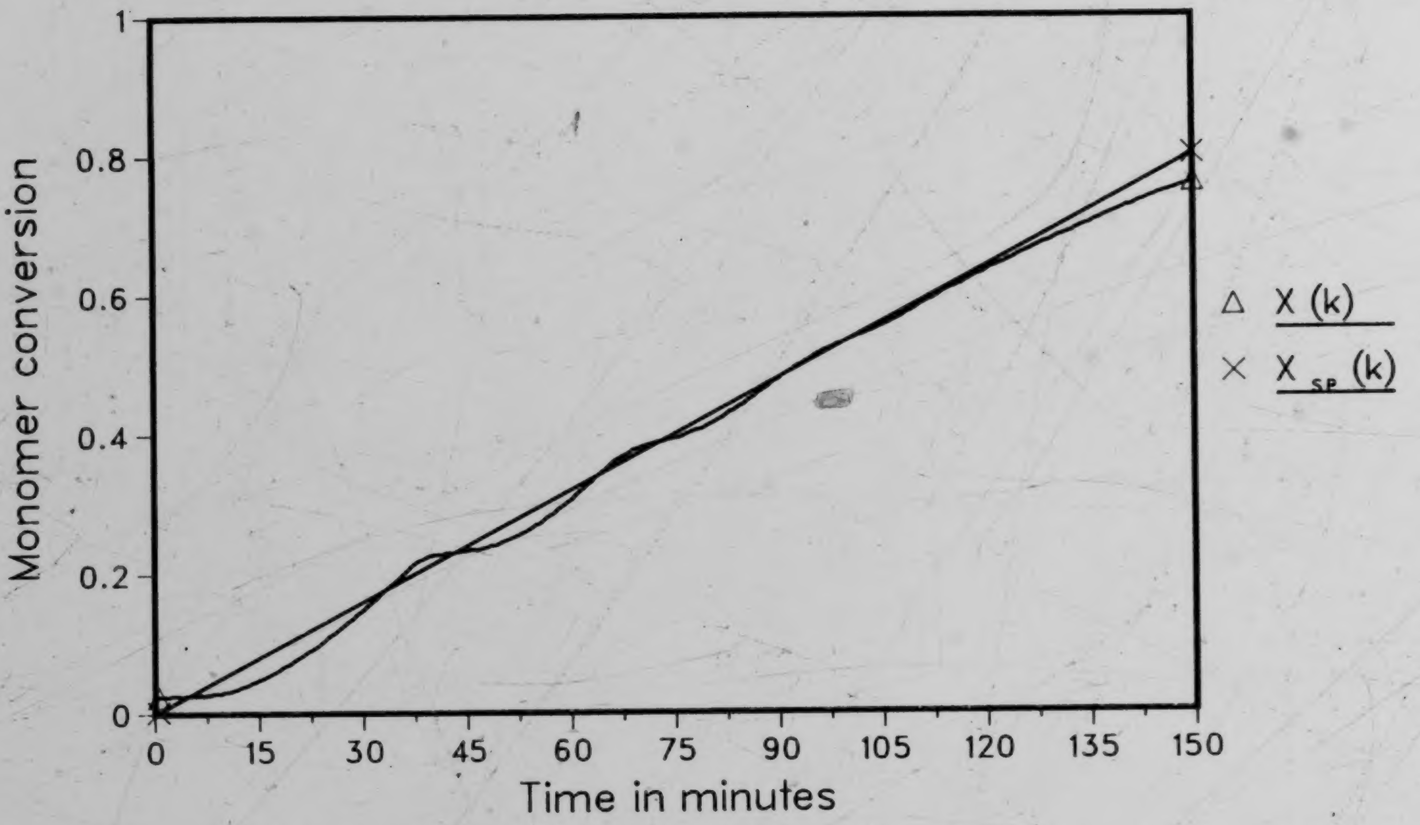


Fig.–5.54 Conversion control with STC
STC/na=3/nb=2/
(zero initial parameters)

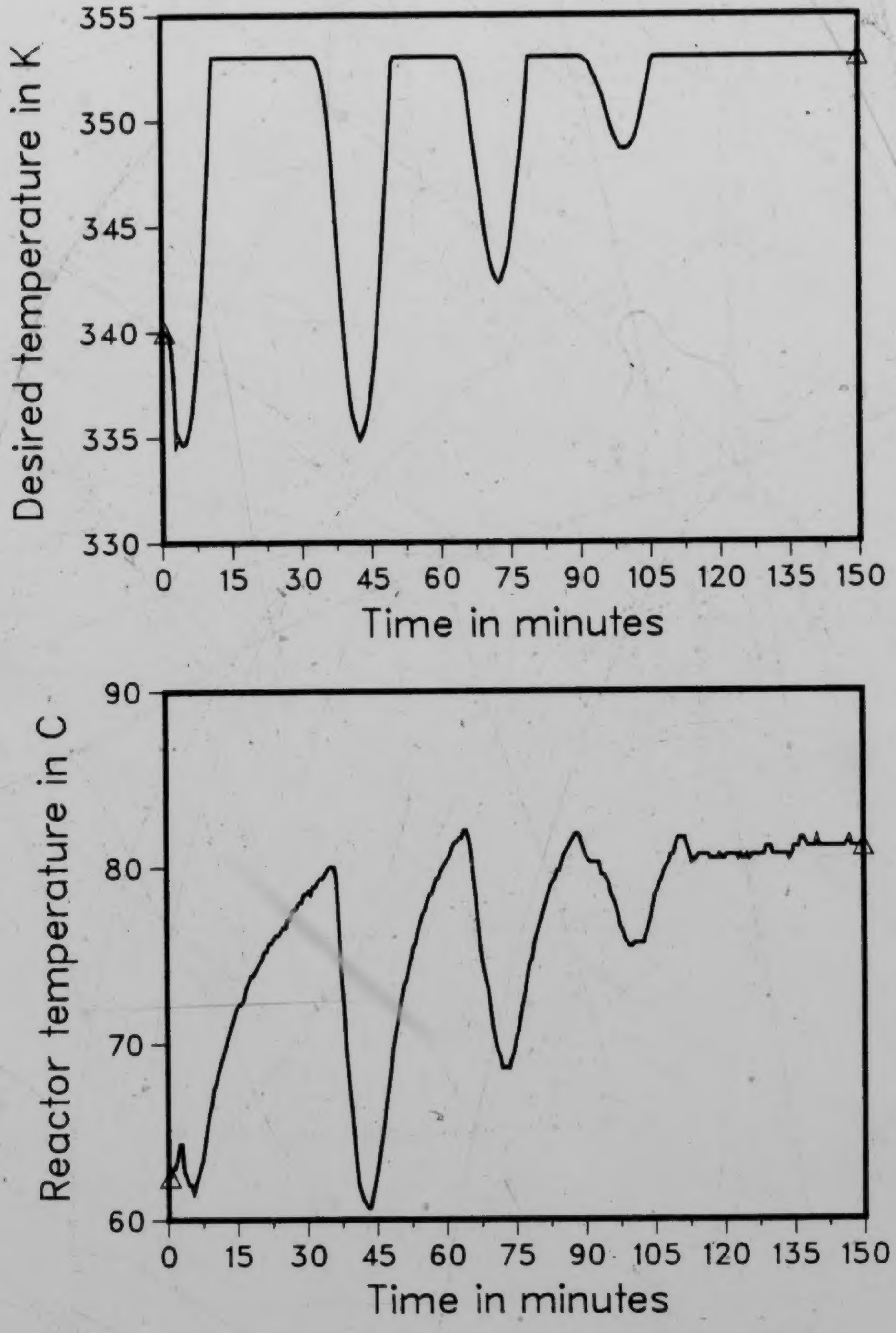
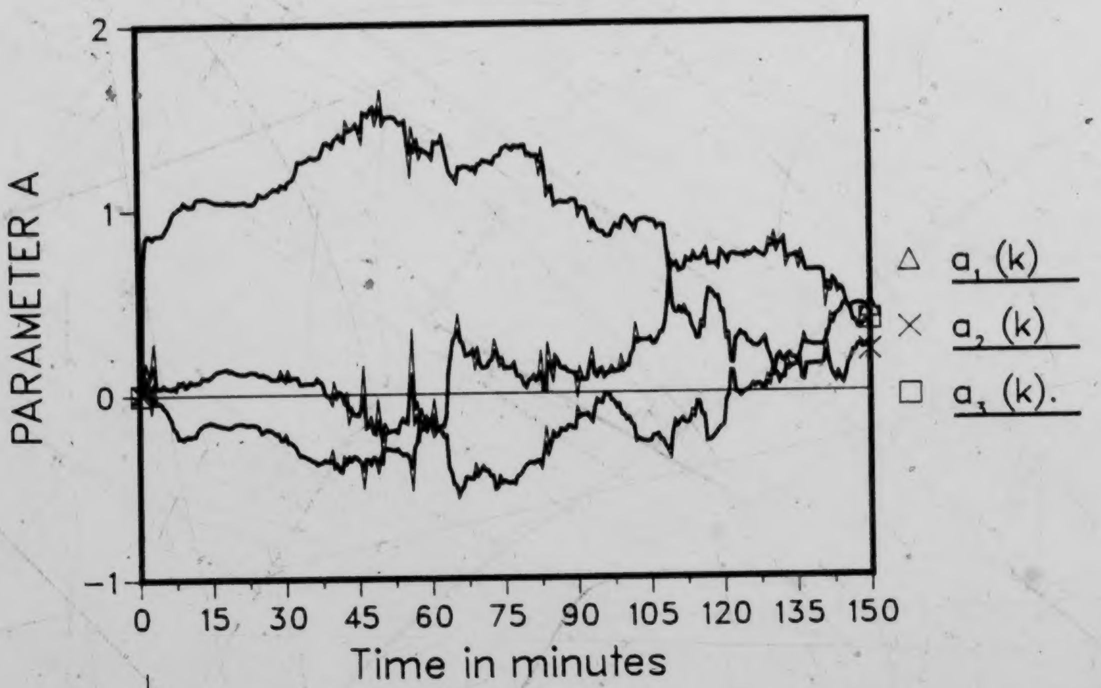


Fig.–5.55 Desired and actual reactor temperature (cf. Fig.–5.54)



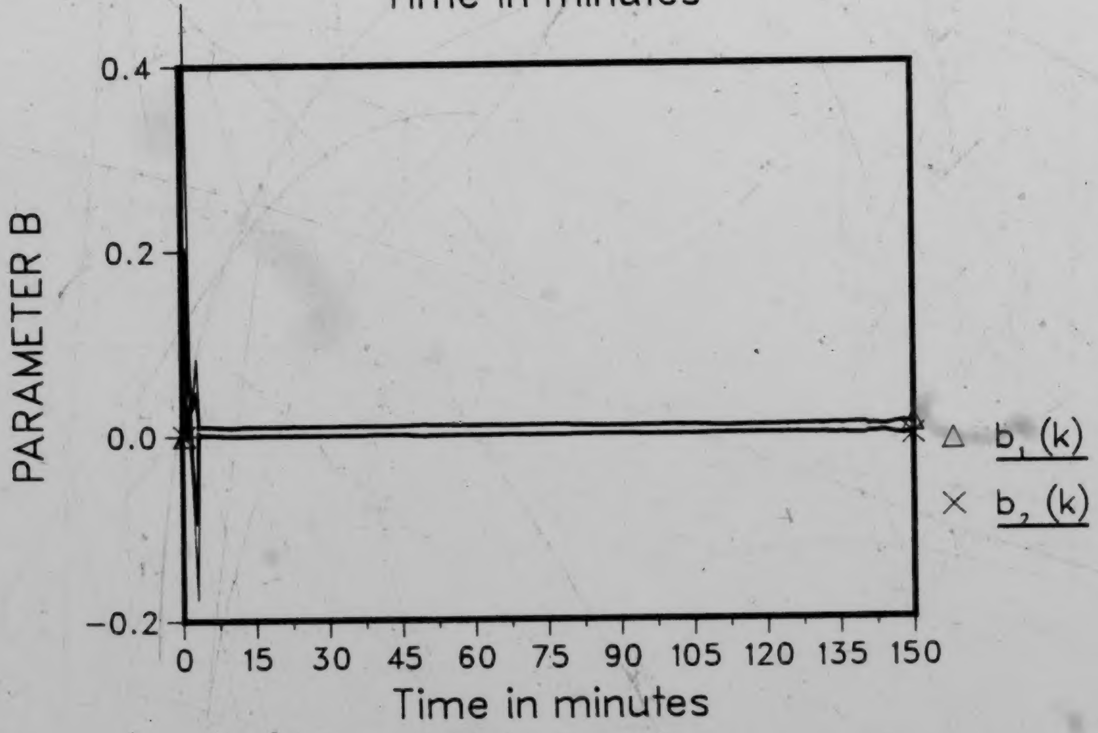


Fig.–5.56 Parameter trajectory of STC
(cf. Fig.–5.54)

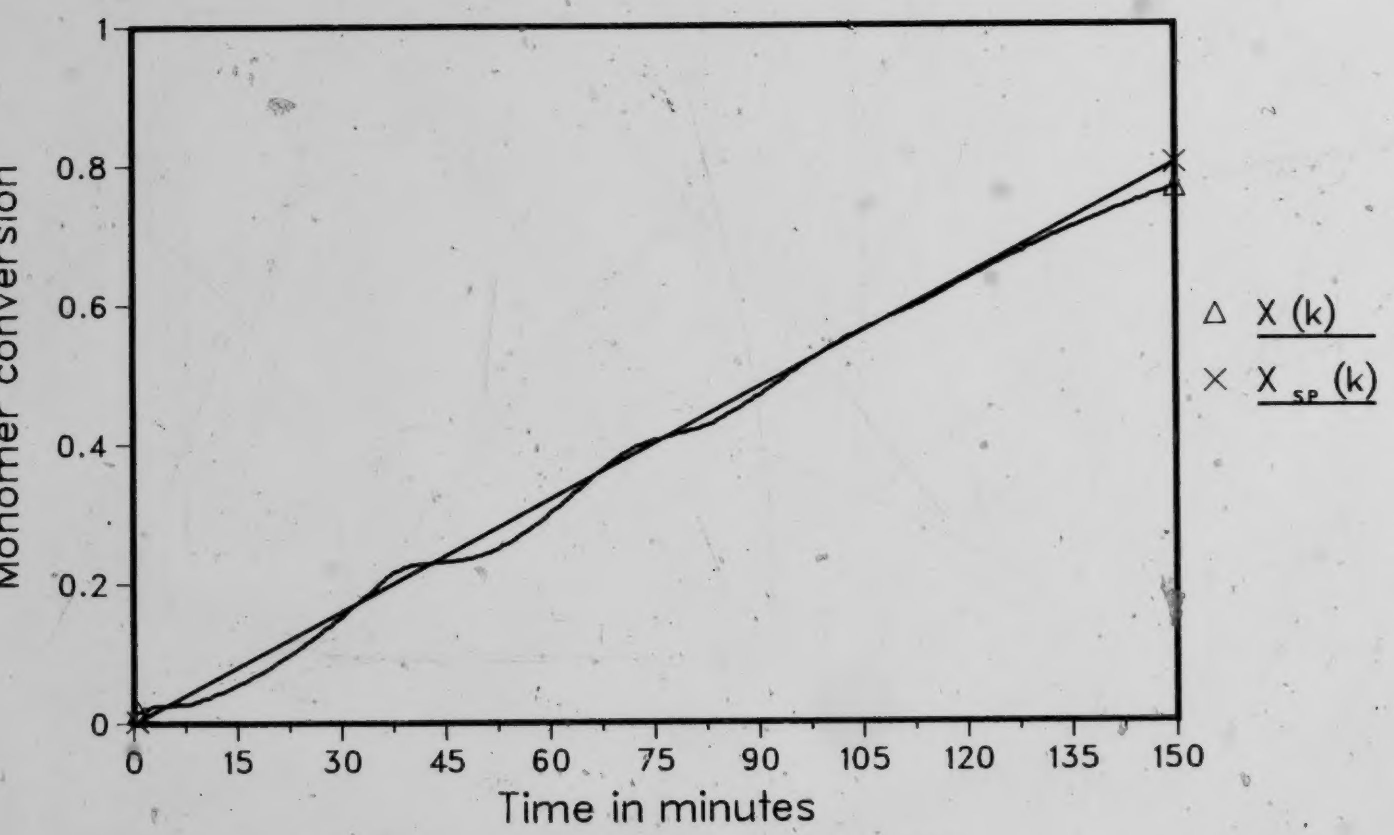


Fig.–5.57 Conversion control with STC
STC/na=2/nb=2/
(zero initial parameters)

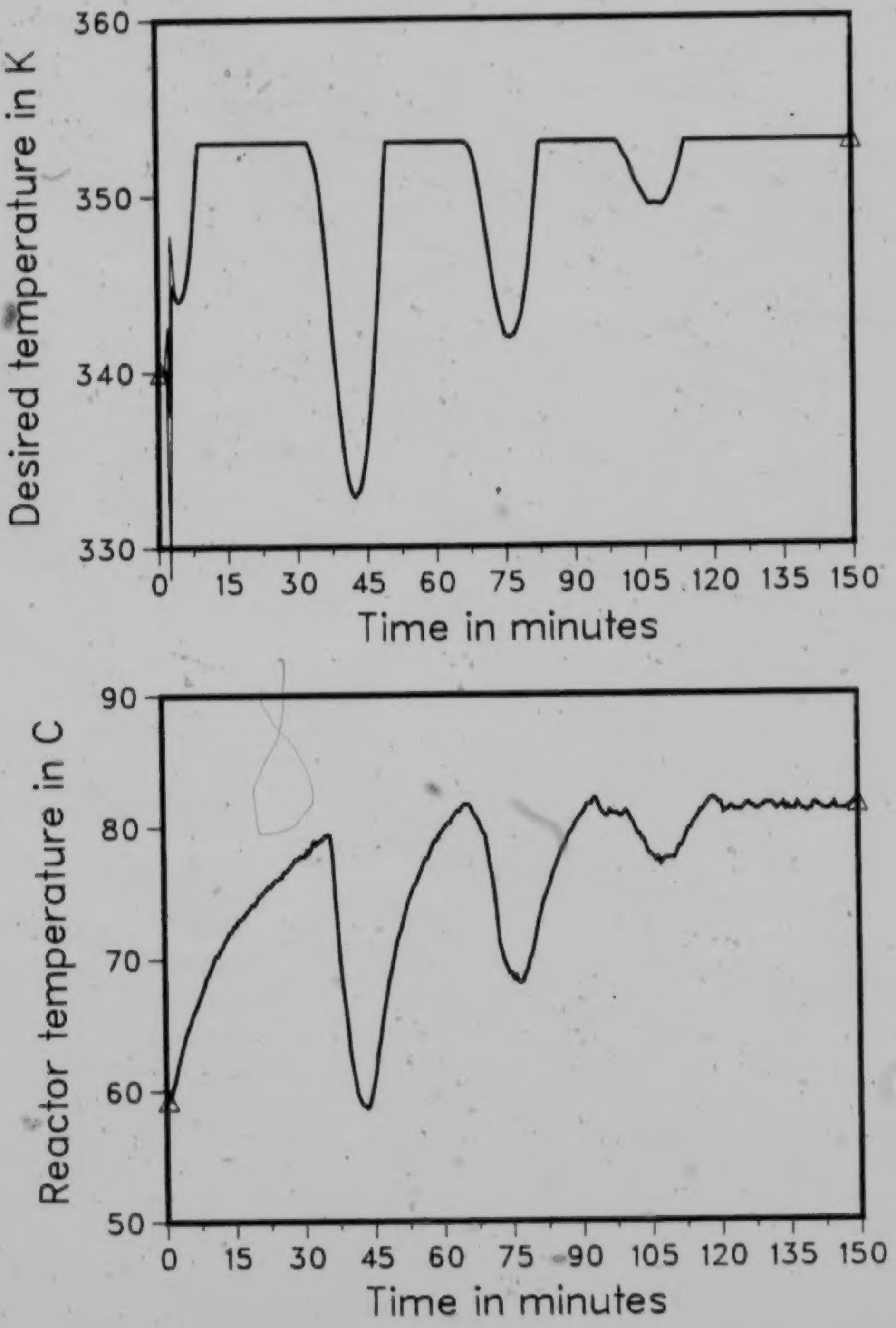


Fig.–5.58 Desired and actual reactor temperature (cf. Fig.–5.57)

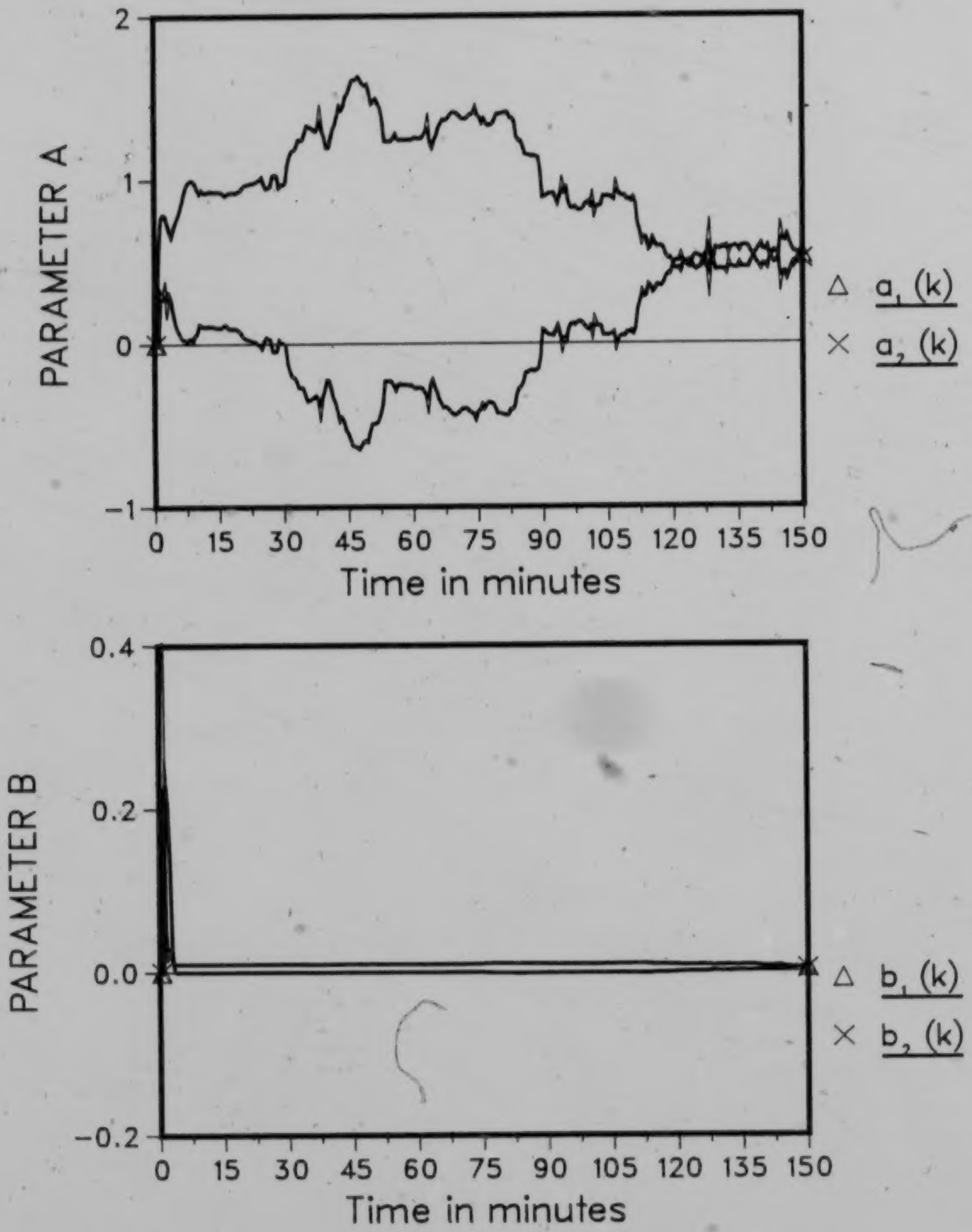


Fig.–5.59 Parameter trajectory of STC
(cf. Fig.–5.57)

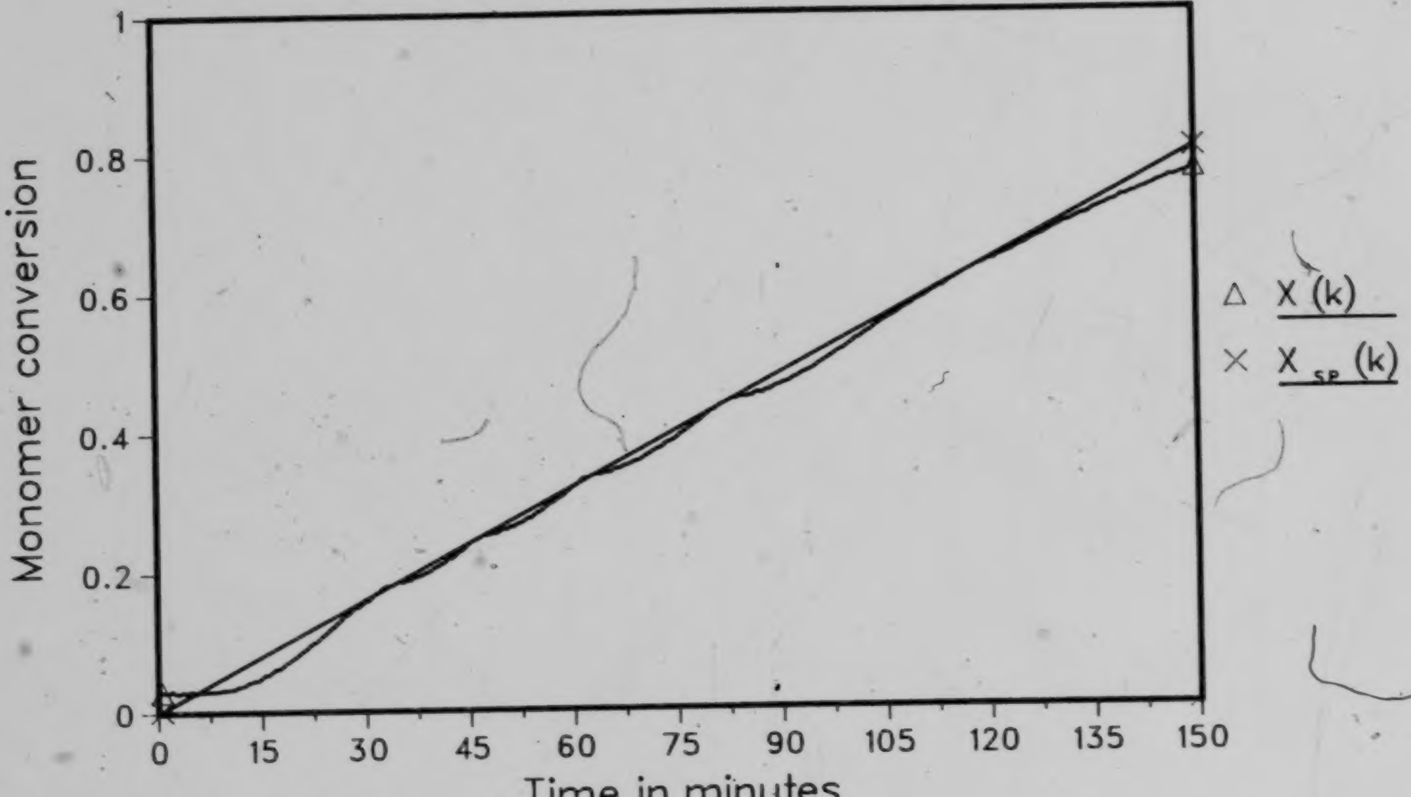


Fig.-5.60 Conversion control with PA
PA/na=3/nb=2/d=0/p=0.5/Uf=340.,
(nonzero initial parameters)

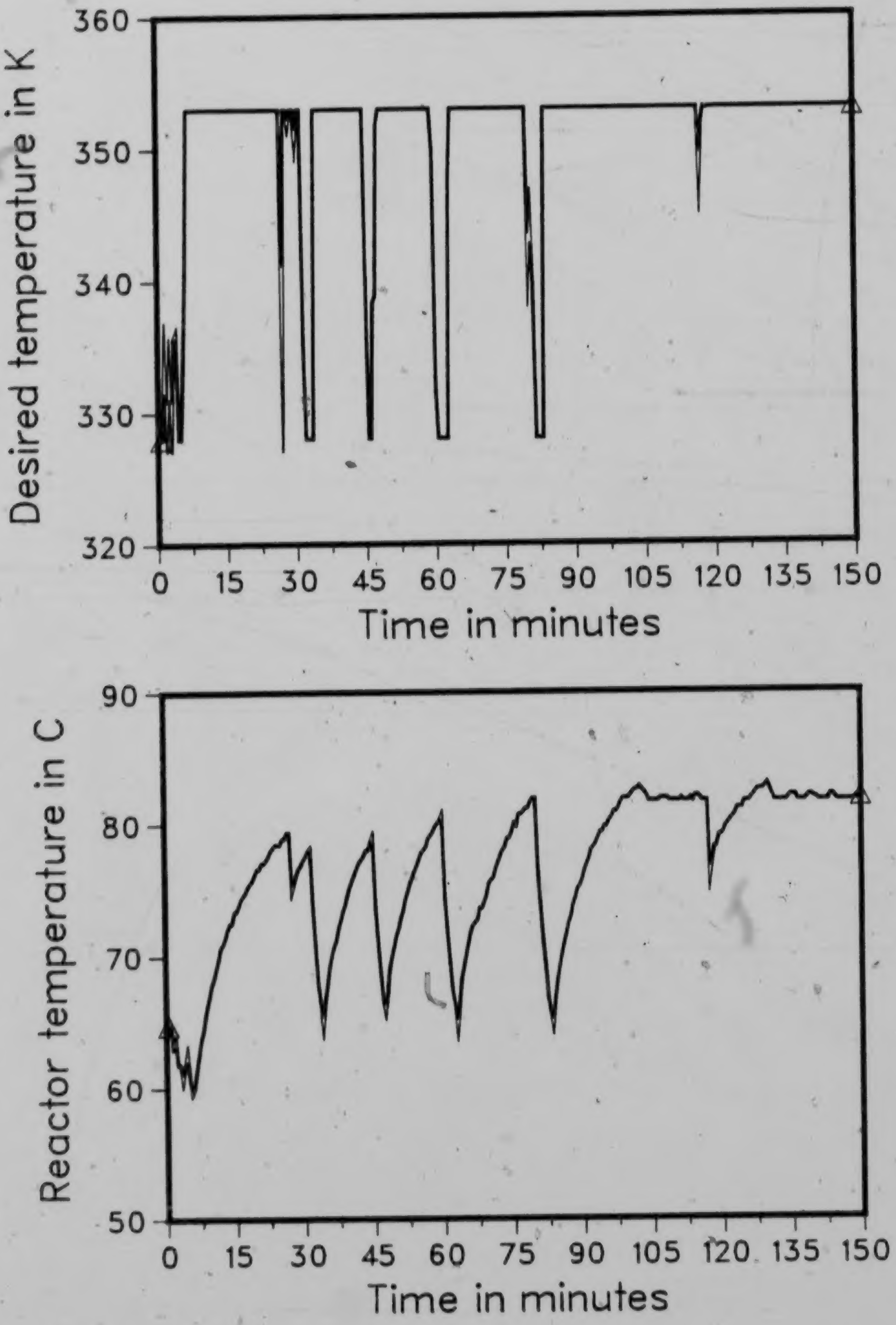


Fig.–5.61 Desired and actual reactor temperature (cf. Fig.–5.60)

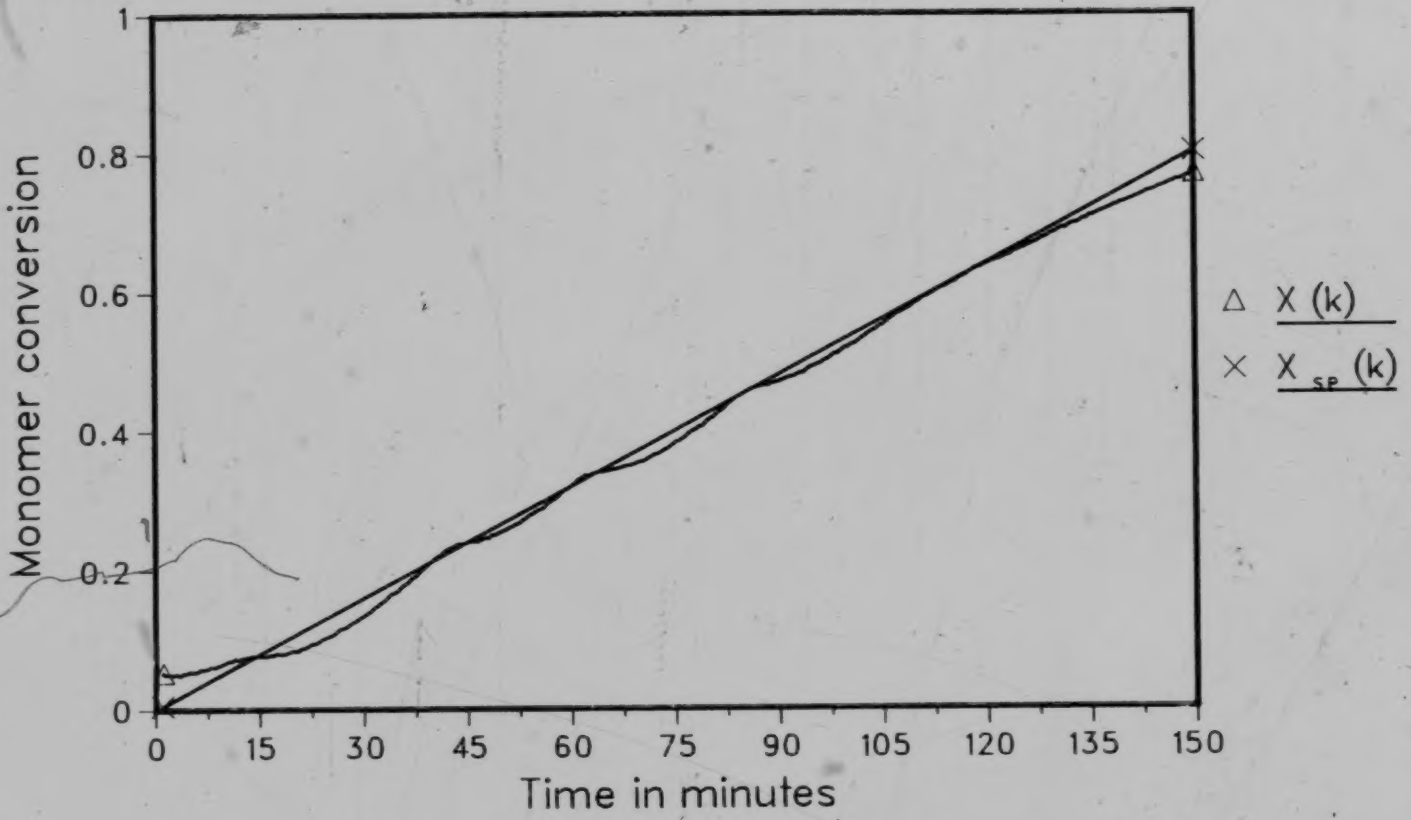


Fig.–5.62 Conversion control with PA
PA/na=3/nb=2/d=0/p=0.5
(nonzero initial parameters)

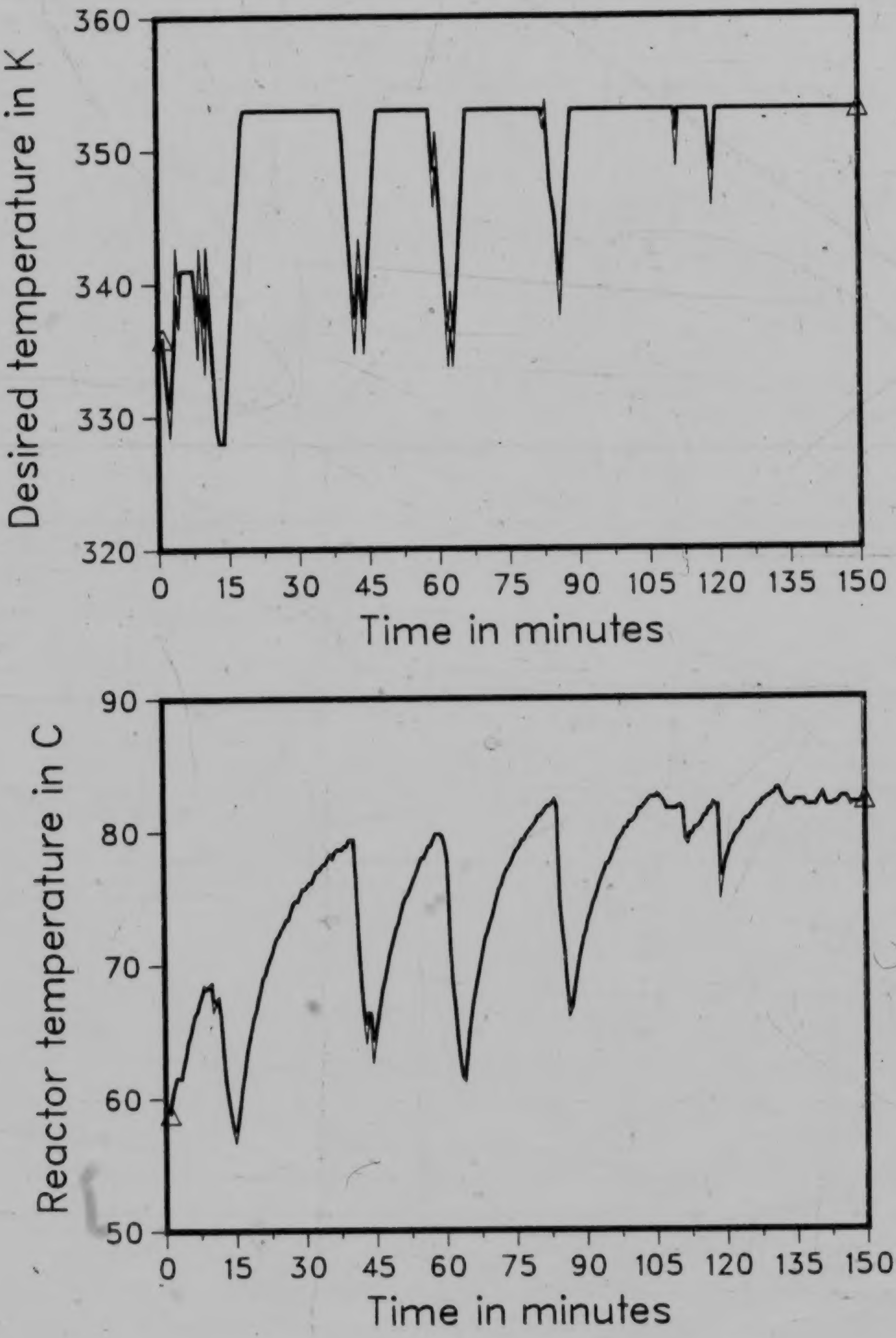


Fig.–5.63 Desired and actual reactor temperature (cf. Fig.–5.62)

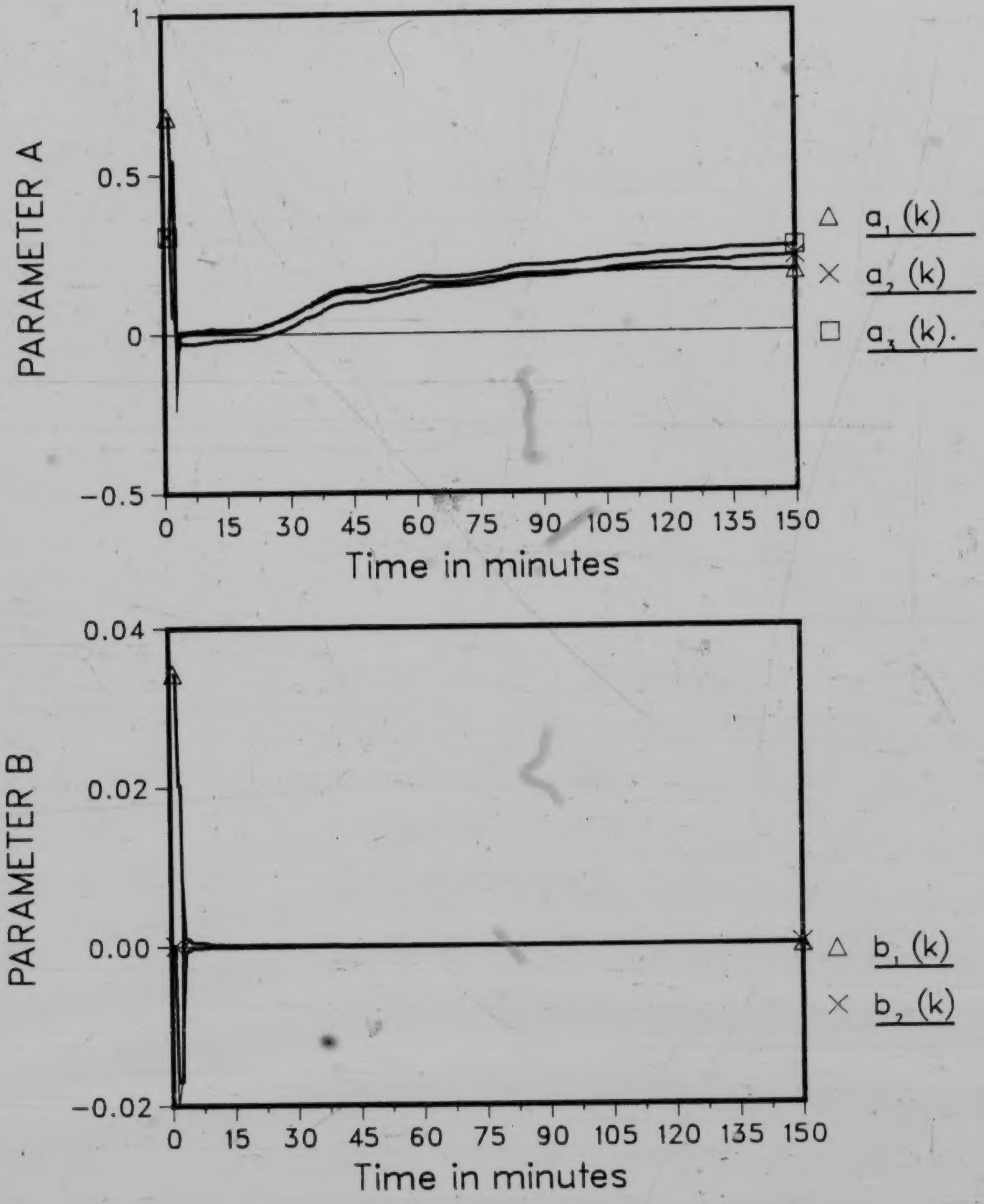


Fig.–5.64 Parameter trajectory of PA
(cf. Fig.–5.62)

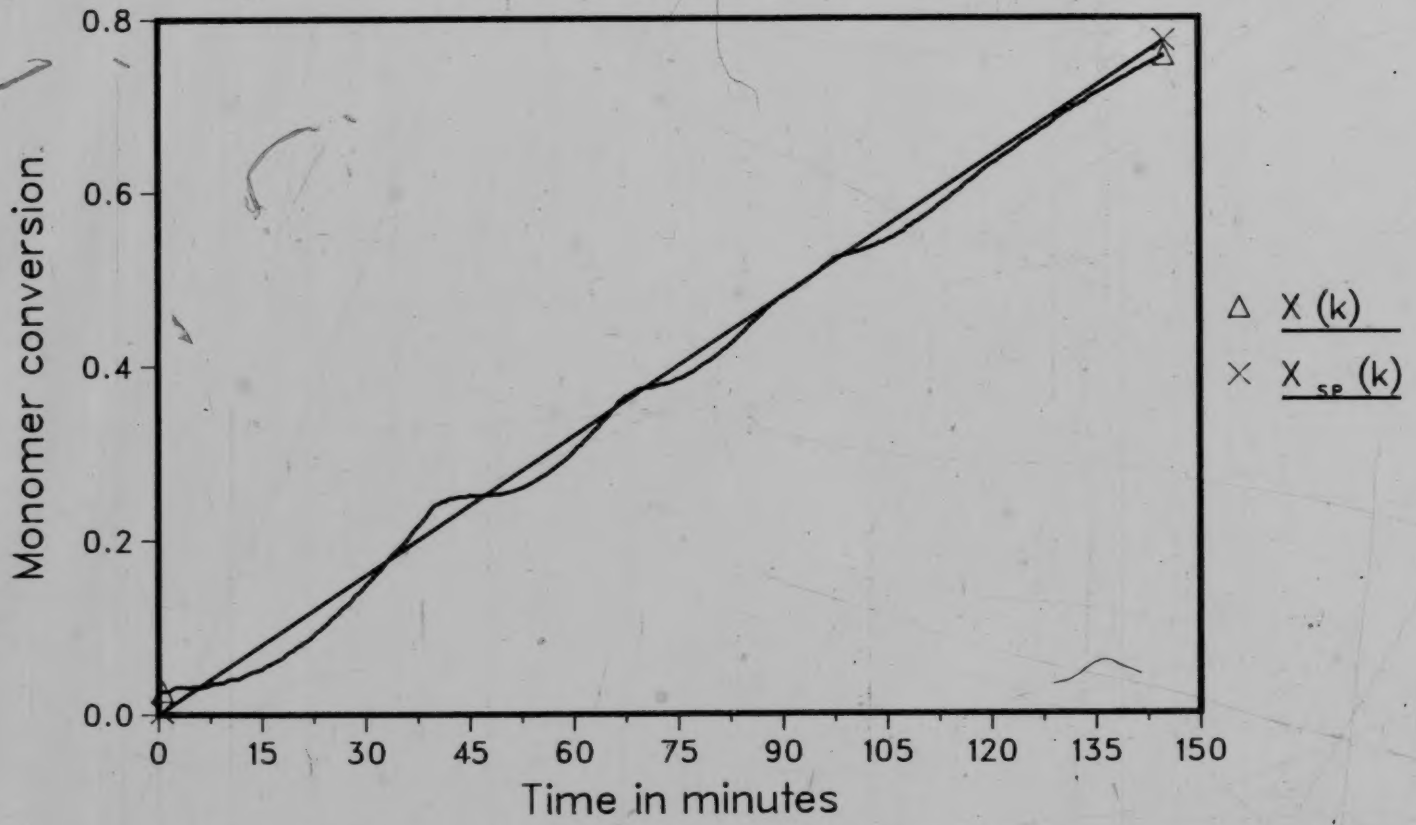


Fig.-5.65 Conversion control with PA
PA/na=3/nb=2/d=0/p=0.5/Uf=340.,
(zero initial parameters)

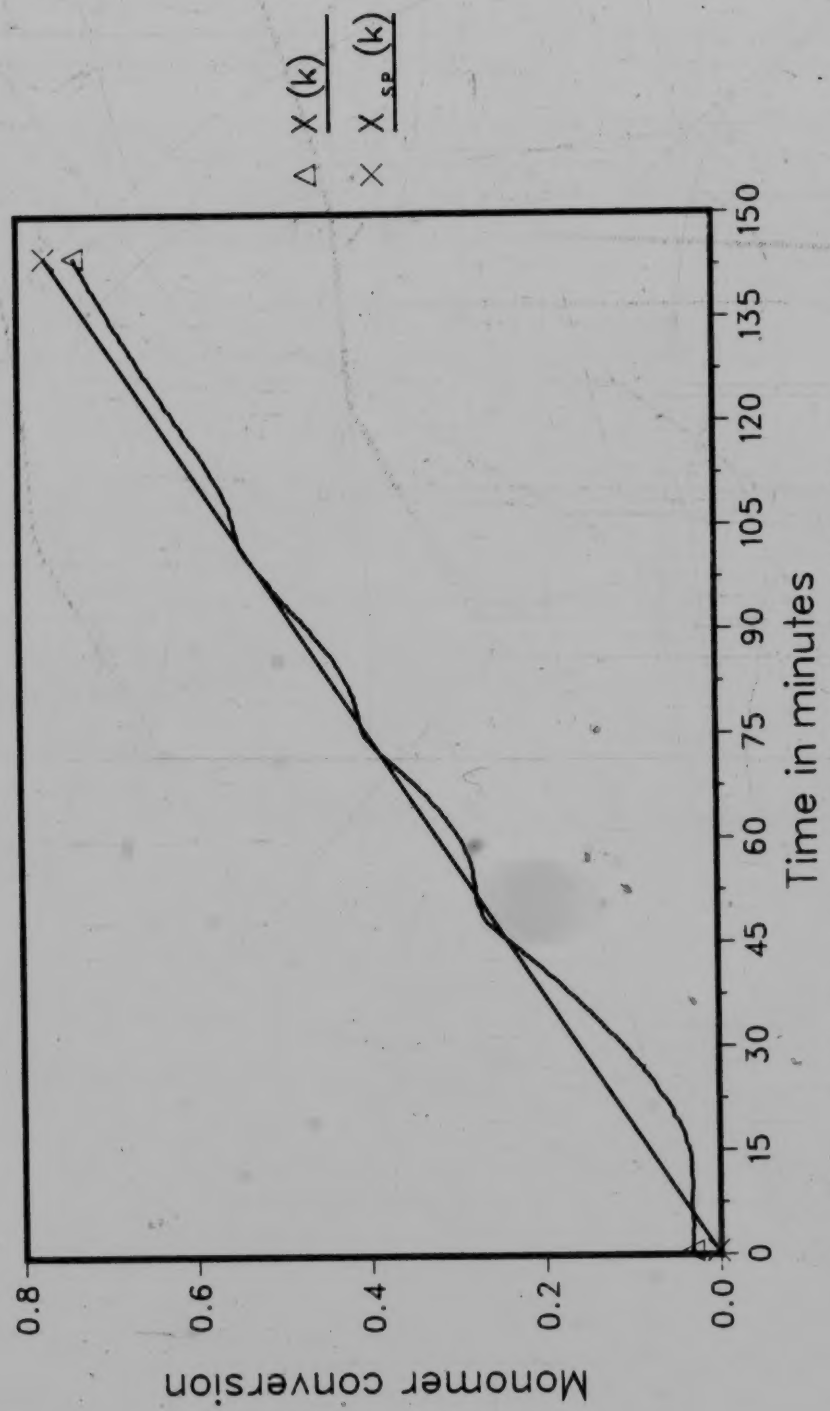


Fig.–5.66 Conversion control with PA
PA/na=2/nb=2/d=0/p=0.5/Uf=340.,
(zero initial parameters)

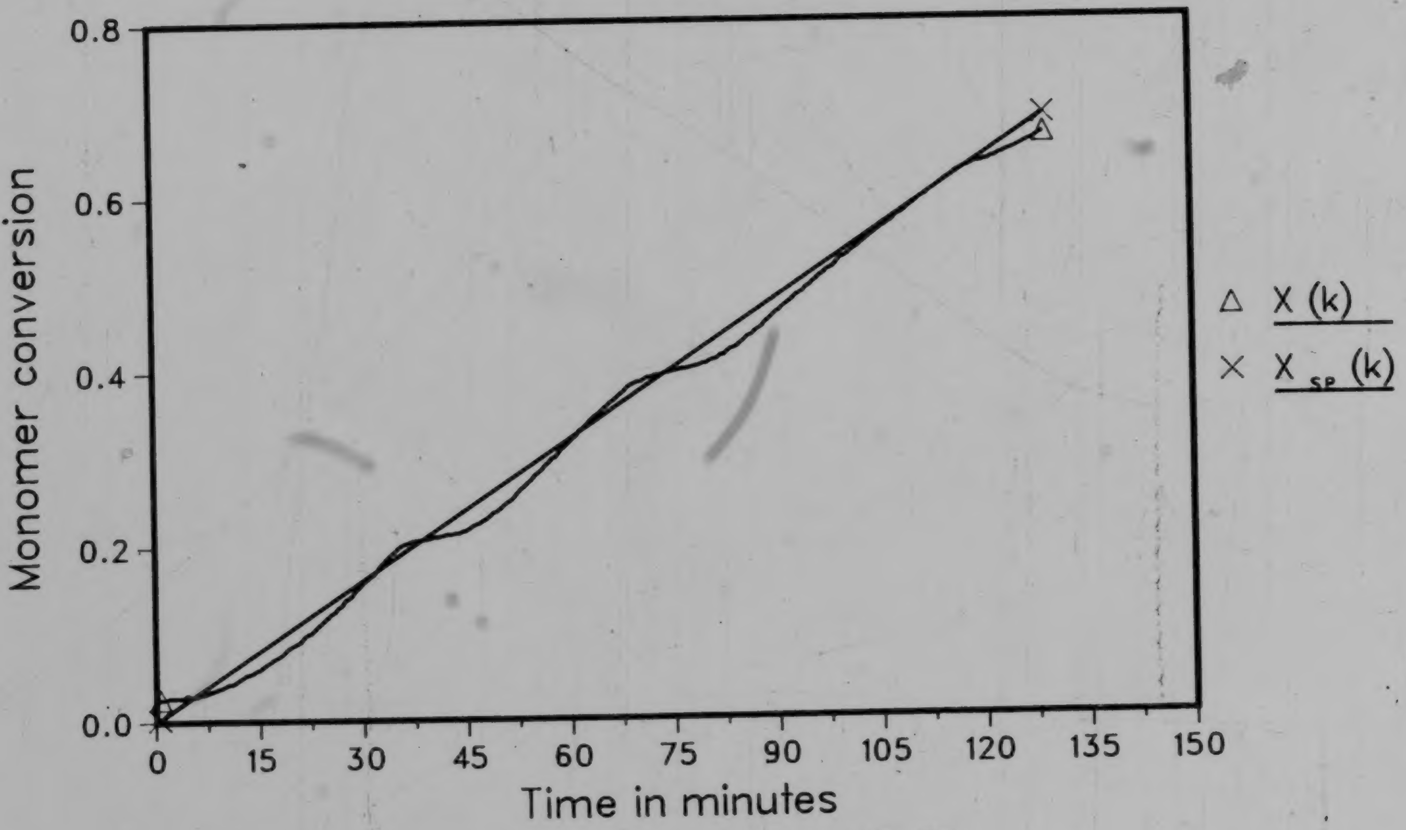


Fig.–5.67 Conversion control with PA
PA/na=2/nb=2/d=0/p=0.5,
(zero initial parameters)

5.7 Conclusions

This chapter is concerned with the applications of adaptive control in the area of polymers. The performance of different control algorithms such as pole-assignment control algorithm, adaptive PID based on pole-assignment, self-tuning controller and fixed gain PID has been examined both experimentally and by simulation studies in a batch polymerization reactor of methylmethacrylate. Monomer conversion and weight average molecular weight control have been studied by simulation studies. Although all the controllers gave excellent control performance, the pole-assignment algorithm seemed to give the best one. Experimental evaluation of the different algorithms was done by controlling the monomer conversion within the reactor. All the control algorithms showed almost the same control performance. An improved control performance would be obtained if the heating dynamics of the system were faster. A problem regarding the parameter estimation was detected when the process input was saturating at its upper or lower limits for long periods of time. The introduction of some excitation into the regressor gave a solution to this problem.

Our main conclusion, based on the results presented in this chapter, is that adaptive control is a promising solution to many control problems but system design considerations have a great influence on its performance. On the other hand, the good performance, in most cases, of a well-tuned PID controller and the need for some type of "tuning" of an adaptive algorithm are the main reasons for the wide use of this simple three term controller in the industry.

6. Conclusions and future work

Adaptive control has been introduced to handle situations where the performance of conventional control strategies was unsatisfactory and insufficient. For adaptive control purposes, a means of mathematical description of the process is needed. The mathematical model must describe the process with the minimum degree of complexity but in a usable form. Knowing the mathematical description of the process, one can then design a control law. The performance of the control algorithm depends on both the design of the control law and the ability of the identification scheme to give "good" estimates of the process parameters (in an explicit control design) or of the controller parameters (in an implicit one).

The main contributions of this work are:

A1] Stability analysis and exponential convergence of the recursive least squares identification scheme with variable forgetting factor.

A2] Stability analysis of a pole-assignment control algorithm in the presence of bounded unmeasured disturbances.

A3] Design of a stable pole-assignment control algorithm with dead time compesation.

A4] Design of a conventional velocity type three term adaptive PID controller.

A5] Experimental evaluation of the pole-assignment algorithm, the self-tuning controller and the fixed gain

PID controller on a batch solution polymerization reactor of methylmethacrylate by controlling the monomer conversion trajectory.

The above conclusions are explained with reference to the relevant chapters in more detail in the following sections.

In chapter 2, the identification scheme used in this work is presented. It is a modification of the original least squares identification algorithm derived by Gauss with a variable forgetting factor. The use of the latter makes the whole algorithm suitable for time-varying processes. It has been shown that this algorithm guarantees boundness and convergence of the parameter estimates as well as minimization of the prediction error. These properties are independent of the boundness of the input/output vector. Unlike Cordero's work (1981), we do not impose any restrictions on the magnitude or bounds on the covariance matrix. The use of a persistently exciting signal (process input for open loop system identification or reference signal for a closed loop system) and the use of an upper bound (less than one) for the forgetting factor make the identification scheme converge to the true process parameters of a time invariant or slowly time-varying process. The convergence of the parameter estimates to the true values is exponentially fast. By using a persistently exciting signal, it has been shown that the covariance matrix remains bounded from above and below for all the time instants. This suggests that the "blow-up" and "drifting" problems can be avoided by using a suitable persistently exciting signal. It was found that there is a relationship between the upper bound of the variable forgetting factor and the amount of excitation applied to the process.

It is very well known that almost all the processes controlled by adaptive algorithms show some short of nonminimum phase behaviour. A minimum phase process can have nonminimum properties for certain choice of sample times. Under feedback control, the closed loop system poles are assigned, in the limit, to the open loop zeros which is not desirable for nonminimum phase systems. Pole-assignment control algorithms with integral action is a particular class of algorithms which avoid the above mentioned problem by shifting the closed loop system poles to desired locations while minimize the steady state error because of the integral action. In chapter 3, a pole-assignment algorithm is presented. Its control design is explicit in nature requiring construction of the control law after the process parameters have been estimated. The closed loop system under its control has been proved stable, despite the presence of bounded disturbances. Although local in nature, the stability analysis is applicable to both minimum and nonminimum phase systems. The possible singularity of the Sylvester matrix, until the estimates of the process parameters converge to some "good" values, is the limiting point of this algorithm. It has been shown that the problem can be eliminated if a fixed gain controller able to stabilize the system and a persistently exciting reference signal are used. In this case, the control algorithm leads to a globally stable closed loop system. A number of simulation examples illustrate the performance of this control algorithm.

In chapter 4, it is shown how the proposed pole-assignment algorithm can become computationally more efficient especially in the presence of large time delays. The idea is to use a Smith predictor in the closed loop system. It has also been shown how the pole-assignment algorithm with the dead time compensation feature leads to a globally stable closed loop system. An adaptive PID controller based on pole-assignment techniques has been derived for a particular class of processes with or without time delays.

In chapter 5, the performance of the proposed pole-assignment algorithm is evaluated both experimentally and by simulation studies on a batch solution polymerization reactor of methylmethacrylate. Its performance is compared with that of a self-tuning controller, a fixed gain PID controller and the adaptive PID controller. Monomer conversion and weight average molecular weight were considered as the process variables under control for simulation purposes. In all the cases, the performance of the pole-assignment algorithm was better than that of the other controllers. Experimentally, monomer conversion has only been studied. Despite the slow heating dynamics of the system, the performance of all the applied control algorithms can be considered as satisfactory. The pole-assignment algorithm performed better than the other controllers only when a set of good initial process parameters was used. The process parameter "drifting" problem, when the process input saturates at its upper or lower limits for long time periods, was avoided by introducing some excitation into the regressor. This experimental study made clear that design problems greatly affect the performance of even the "best" control algorithm.

An abudance of ideas for future work arise from this thesis. Having seen the usefulness of persistently exciting signals in adaptive control, we should improve our knowledge about them. Research should be devoted on the design of suitable persistently exciting signals. Since limited amount of excitation is allowed in a closed loop system, it is very desirable to obtain the true process parameters by using minimum excitation. Because of it, two questions should be answered: what is the minimum amount of excitation required for the identification of a specific number of process parameters and at what frequencies should this excitation occur. The pole-assignment algorithm would become computationally more efficient i.e. avoid checks on the singularity of the Sylvester matrix if it had an implicit control law. An interesting problem is how this can be done without sacrificing the stability properties of the explicit algorithm. Of course, an interesting experimental follow-up can be the weight average molecular weight control in the batch solution polymerization reactor. The use of a split range valve will improve the dynamics of the experimental system during the heating and cooling periods. This leads to an interesting control problem when adaptive algorithms are applied because two different models are required for the description of the polymerization process.

The list of ideas for future work can go on and on.

7. Nomenclature

7.1 Technical Abbreviations

DC	Determinisitic Case
GC	Gas Chromatography
PA	Pole-Assignment
PASP	Pole-Assignment with Smith Predictor
PID	Proportional-Integral-Derivative controller
SC	Stochastic Case
STR	Self-Tuning Regulator
STC	Self-Tuning Controller

7.2 Alphabetic

$A(z^{-1})$	Polynomial corresponding to the process output
$\hat{A}(z^{-1},k)$	Estimate of polynomial $A(z^{-1})$ at time k
a_i's	Coefficients of polynomical $A(z^{-1})$
$\hat{a}_i(k)$'s	Estimates of a_i's at time k
A_i's	Frequency factors
$A^*(z^{-1})$	Desired closed loop polynomial
a_i^*'s	Coefficients of polynomial $A^*(z^{-1})$
$B(z^{-1})$	Polynomial corresponding to the process input
$\hat{B}(z^{-1},k)$	Estimate of polynomial $B(z^{-1})$ at time k
$B_1(z^{-1})$	Polynomial related to $B(z^{-1})$ by: $B(z^{-1})=z^{-d}B_1(z^{-1})$
b_i's	Coefficients of polynomial $B(z^{-1})$

$\hat{b}_i(k)$'s	Estimates of b_i's at time k
C	Constant parameter in equation (3.47)
C_1, C_2	Positive constant parameters
Cf	Parameter equal to $2f(1-k_{ts}/k_t)$
d	Process time delay
d_s	Number of zeros on the unit circle
E_i's	Activation energies
E{}	Statistical expectation operator
e(k)	Prediction error
f	Initiator efficiency
g	Upper bound of parameter error vector norm at time k=0
$\underline{I}$	Identity matrix
[I]	Initiator concentration
J_1	Loss function in equation A.5
$J_k(\underline{\theta})$	Loss function in equation (2.12)
$\underline{K}(k)$	Parameter estimator gain
K_1	Parameter equal to $A_1\exp(-E_1/RT)$
K_2	Parameter equal to $A_2\exp(-E_2/RT)$
K_3	Parameter equal to $A_3\exp(-E_3/RT)$
K_4	Parameter equal to $A_4\exp(-E_4/RT)$
K_5	Parameter equal to $A_5\exp(-E_5/RT)$
K_6	Parameter equal to $A_6\exp(-E_6/RT)$
k_1	Condition number of $\underline{P}(0)^{-1}$
k_2, k_3	Positive parameters in equation (3.53)
K_m	Proportional gain
k_d	Dissociation rate constant

k_i	Initiation rate constant
k_m	Transfer to monomer rate constant
kp	Propagation rate constant
k_s	Transfer to solvent rate constant
k_t	Overall termination rate constant
k_t	Termination by combination rate constant
k_{td}	Termination by disproportionation rate constant
$L(z^{-1})$	Polynomial in the denominator of the pole-assignment control law
$\hat{L}(z^{-1},k)$	Estimate of polynomial $L(z^{-1})$ at time k
l_i's	Coefficients of polynomial $L(z^{-1})$
$\hat{l}_i(k)$'s	Estimates of the coefficients l_i's at time k
M	Parameter in equation (3.48)
[M]	Monomer concentration
M_n	Number average molecular weight
Mw	Weight average molecular weight
Mw_s	Weight average molecular weight set point
M_x	Molecular weight of polymer molecule with chain lenght x
m	Order of polynomial $B_1(z^{-1})$
N	Asymptotic merory length
N_0	Parameter in equation (2.23)
N_x	Mole fraction of polyner molecules with chain length x
n	Order of polynomial $A(z^{-1})$
n(k)	Bounded disturbance at time k
$\underline{P}(k)$	Covariance matrix at time k

$P(z^{-1})$	Polynomial in the numerator of the pole-assignment control law
P_x	Polymer molecules with chain length x
P_i's	Coefficients of polynomial $P(z^{-1})$
$\hat{p}_i(k)$'s	Estimates of the coefficients p_i's at time k
R	Constant of ideal gases
$R\cdot$	Free radicals formed by the initiator dissociation
R_x	Free radicals with chain length x
r	Parameter equal to max $\{n,m+d\}$
S	Solvent molecules
s	Time interval of persistent excitation
T	Temperature of reaction
tf	Final time of reaction
u(k)	Process input at time k
$u_i(k)$	Incremental process input at time k
V(k)	Lyaponov type function at time k defined in equation (2.41)
V_c	Limiting value of V(k)
W(k)	Function defined in equation (2.52)
w(k)	Auxiliary signal at time k defined in equation (3.39)
w_x	Weight fraction of polymer molecules with chain length x
x	monomer conversion
xf	Final monomer conversion
x(k)	Auxiliary signal at time k defined in equation (3.41)

$y(k)$	Process output at time k
$\hat{y}(k)$	Estimated process output at time k
$y_i(k)$	Incremental process output at time k
$y_s(k)$	Auxiliary process output at time k
$y^*(k)$	Desired reference signal
z^{-1}	Time delay operator

7.3 Greek

$\alpha(z^{-1})$	Polynomial in z^{-1} defined in equation A.2
$\beta(z^{-1})$	Polynomial in z^{-1} defined in equation A.3
$\gamma(z^{-1})$	Polynomial defined in equation (3.48a)
γ_i's	Coefficients in equation (3.47)
$\delta(z^{-1})$	Polynomial defined in equation (3.49a)
δ_i's	Coefficients of polynomial $\delta(z^{-1})$
Δ_m	Lower limit of the upper bound of the noise
Δb	Maximum upper bound of the noise
$\epsilon(k)$	Tracking error
ϵ_i's	Coefficients in equation (3.50)
ζ	Weigthing factor in the self-tuning controller
η	Norm of the vector formed by the coefficients δ_i's
$\underline{\theta}_0$	True process parameter vector
$\underline{\theta}(k)$	Estimate of $\underline{\theta}_0$ at time k
λ	Constant forgetting factor
$\lambda(k)$	Variable forgetting factor at time k
λmax	Maximum value of the forgetting factor $\lambda(k)$

$\bar{\lambda}max$	Maximum eigenvalue of a matrix
λmin	Minimum value of the forgetting factor $\lambda(k)$
$\bar{\lambda}min$	Minimum eigenvalue of a matrix
$\mu(z^{-1})$	Polynomial defined in equation (3.52)
μ	Parameter equal to $\{\lambda_{min}[\underline{P}(0)^{-1}]\}^{-1}$
μ_0	Zeroth moment of dead polymer
μ_2	Second moment of dead polymer
ν	Parameter defined in equation (3.56)
ξ	Parameter in equation (3.48)
π_i's	Coefficients in equation (3.50)
ρ	Reaction mixture density
ρ_1 to ρ_4	Positive constants
$\rho(k)$	Stopping criterion for the parameter adaptation
Σ_0	Parameter defined as: $\Sigma_0=\sigma_0^2 N_0$
σ_0^2	Noise variance
σ_1, σ_2	Positive constants in equation (3.54)
τ_d	Derivative gain
$\hat{\tau}_d$	Dimensionless derivative gain
τ_i	Integral gain
$\hat{\tau}_i$	Dimensionless integral gain
τ_s	Sampling time
$\Phi(k)$	Augmented process output defined in equation A.5
Ω	Parameter in equation (3.48)

7.4 Superscripts

$\hat{\ }$	Estimated value
^{t}	Matrix transpose

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9. Appendix: Basics about STR/STC control

The principle theoretical developments of self-tuning regulators are discussed in Aström and Wittenmark (1973), Wittenmark (1974), Ljung and Wittenmark (1974), Clarke and Gawthrop (1975).

In this part, a brief decription of the self-tuning controller used in this work will be presented.

Assume a discrete, time invariant, single input/ single output process described by the following ARMA model

$$y(k+d) = \alpha(z^{-1})\, y(k) + \beta(z^{-1}) \nabla^{d_s} u(k) + \epsilon(k+d) \qquad (A.1)$$

where

$$\alpha(z^{-1}) = \alpha_0 + \alpha_1 z^{-1} + \ldots + \alpha_n z^{-n} \qquad (A.2)$$

$$\beta(z^{-1}) = \beta_0 z^{-1} + \beta_2 z^{-1} + \ldots + \beta_m z^{-m} \qquad (A.3)$$

d is the number of whole periods of delay and it equals one plus the integer portion of the transport delay divided by the sampling interval. If this parameter is less than the true number of whole periods of delay, there will be a rapid degradation in the ability to control the process. In cases where d is unknown, overestimation rather than underestimation is preferable (Wittenmark (1974)). ∇ is the backwards shift operator and equals to $1-z^{-1}$. The parameter d_s is characteristic of the type of disturbances (stationary or nonstationary). A value of d_s greater than zero always naturally leads to integral action in the controller. The term $\epsilon(k+d)$ is the moving average error and is assumed to be uncorrelated with the regressor variables $\{y(k)\ \ y(k-1)\ \ \ldots\ \ u(k)\ \ u(k-1)\ \ldots\ \}$. The variable $y(k)$ is defined as the deviation of the process output from its desired value while $u(k)$ is the deviation of the process input from its steady state value. This steady state value need not be known if d_s is greater than zero.

The minimum variance control is given by

$$\nabla^{d_s}u(k) = -\frac{\hat{\alpha}(z^{-1})}{\beta(z^{-1})}\,y(k) \qquad (A.4)$$

The parameter vector

$$\underline{\theta}(k)^t = [\ \hat{\alpha}_0\ \ldots\ \hat{\alpha}_n\ \beta_0\ \ldots\ \beta_m]$$

is updated at every sampling interval and is used in the control law (A.4) instead of the true parameters. A number of identification algorithms (least squares in our case) can be used to solve the parameter estimation problem.

The minimun variance control law (A.4) can create excessive or large variations in the manipulated variable. Instead of minimizing the variance of the process output, Clarke and Gawthrop (1975) developed the self-tuning controller by minimizing the loss function

$$J_1 = E\{[y(k+d) + \zeta \nabla^{d_s} u(k)]^2 = E\{\Phi(k+d)^2\} \qquad (A.5)$$

where E stands for the expectation operator.
Using the definition of $\Phi(k+d)$ from (A.5) it can be shown that a self-tuning algorithm with estimation model

$$\Phi(k+d) = \alpha(z^{-1})\, y(k) + \gamma(z^{-1})\, \nabla^{d_s} u(k) + \epsilon(k+d) \qquad (A.6)$$

and the control law

$$\nabla^{d_s} u(k) = - \frac{\hat{\alpha}(z^{-1})}{\gamma(z^{-1})}\, y(k) \qquad (A.7)$$

will have the same properties as derived by Aström and Wittenmark (1973) for the minimum variance case. The order of $\gamma(z^{-1})$ may be larger than the order of $\beta(z^{-1})$ in the uncontrained case. The parameter ζ is a weighting factor for the variance of the manipulated variable in the criterion J_1. Large values for the parameter ζ allow small variations in the manipulated variable while small values allow the opposite. The best value for the constraining factor ζ is unknown but can be easily found by on-line tuning.